Reg. No .... Name **26P254**

M. Sc. DEGREE END SEMESTER EXAMINATION - APRIL 2026

SEMESTER 2 : MATHEMATICS

COURSE : **24P2MATT09 : ALGEBRAIC NUMBER THEORY**

(For Regular 2025 Admission and Improvement/Supplementary 2024 Admission)

Time : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. True or false : If a non zero ideal $\mathfrak{a}$ is prime, then $\mathrm{N}(\mathfrak{a})$ is a prime. Justify. (A, CO 4)
2. Let D be a domain and x and y non-zero elements of D. Prove that $x|y$ if and only if $\langle x\rangle \supseteq \langle y\rangle$. (A, CO 3)
3. Show that if K is a number field, then $K=\mathbb{Q}(\theta)$ for some algebraic integer θ. (A, CO 1)
4. Prove that factorization into irreducible is possible in $\mathfrak{O}$. (A, CO 3)
5. Compute the group of units of a field. (U, CO 3)
6. Prove or disprove: K-conjugates of α are distinct. (A, CO 1)
7. Show that integral basis $\{1, \sqrt{d}\}$ of $\mathbb{Q}(\sqrt{d})$ has the discriminant $4d$ if $d \neq 1(\bmod 4)$. (A, CO 2)
8. Prove that $\mathbb{R}[x, y]/\langle x\rangle$ is isomorphic(as rings) to $\mathbb{R}[y]$. (A, CO 4)
9. Define quadratic number field. (U, CO 2)
10. Find the discriminant of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$. (A, CO 1)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Let $\mathfrak{a} \neq 0$ be an ideal of $\mathfrak{O}$. Prove that there exist ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ such that $\mathfrak{p}_1 \ldots \mathfrak{p}_r \subseteq \mathfrak{a}$. (An, CO 4)
12. Show that the set of all units in a ring with unity forms a group with respect to multiplication. (A)
13. Let $K=\mathbb{Q}(\theta)$ be a number field where θ has minimum polynomial p of degree n. prove that the $\mathbb{Q}$-basis $\left\{1, \theta, \theta^2, \ldots, \theta^{n-1}\right\}$ has discriminant $\Delta\left[1, \theta, \cdots \theta^{n-1}\right]=(-1)^{n(n-1)/2} N\left(D_P(\theta)\right)$ (An, CO 2)
14. Prove that every Euclidean domain is a unique factorization domain. (An, CO 3)
15. Define the field of quotients of an integral domain. Prove that field of quotient of $\mathfrak{O}_K$ is K. (A, CO 4)
16. Show that $\mathbb{Q}(\alpha)=\mathbb{Q}(\theta)$, if and only if all K-conjugates of α are distinct. (A, CO 1)
17. Let $K=\mathbb{Q}(\theta)$ be a number field of degree n. Show that the elements $\sigma_i(\theta)=\theta_i$ are the distinct zeros in $\mathbb{C}$ of the minimum polynomial of θ over $\mathbb{Q}$. (An, CO 1)
18. Let $K=\mathbb{Q}(\zeta)$ where $\zeta=e^{2\pi i/5}$. Calculate $N_K(\alpha)$ and $T_K(\alpha)$for $\alpha=\zeta+\zeta^2$. (An, CO 2)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Prove that algebraic number α is an algebraic integer if and only if minimum polynomial for α has coefficients in $\mathbb{Z}$. (An, CO 1)
20. Compute trace of a general element in $\mathbb{Q}(\zeta)$. (A, CO 2)
21. Prove that the non-zero fractional ideals of $\mathfrak{O}$ form an abelian group under multiplication. (An, CO 4)
22. Define Euclidean quadratic Field. Prove that the ring of integers $\mathfrak{O}$ of $\mathbb{Q}(\sqrt{d})$ is Euclidean for $d = -2, -11$. (U, CO 3)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Explain the properties of algebraic numbers, algebraic integers and integral bases.	U	3, 6, 10, 16, 17, 19	12
CO 2	Distinguish quadratic and cyclotomic extensions and traces and norms.	A	7, 9, 13, 18, 20	11
CO 3	Analyze factorization into irreducibles in Euclidean Domains and quadratic fields	An	2, 4, 5, 14, 22	10
CO 4	Analyze prime factorization of Ideals, the norm of an ideal, nonunique factorization of cyclotomic fields	An	1, 8, 11, 15, 21	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;