Reg. No .... Name **26P404-S**

M.Sc. DEGREE END SEMESTER EXAMINATION- MARCH 2026

SEMESTER 4 : MATHEMATICS

COURSE : **21P4MATTEL16 : SPECTRAL THEORY**

(For Supplementary 2023/2022/2021 Admissions)

Time : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. (a) Let F be a closed subspace of a Hilbert space H. For $x \in H$, define the orthogonal projection of x on F, and the orthogonal complement of F. (b) Explain the complemented subspace property. (U, CO 3)

2. Let X and Y be $nls's$ and $F: X \to Y$ be linear. If F is compact and U is the open unit ball in X, show that $F(U)$ is totally bounded. Show that the converse holds if Y is Banach. (E, CO 1)

3. State the Riesz representation theorem. (U, CO 3)

4. If (x_n) is a bounded sequence in a normed linear space X and for some subset S of X' with $\overline{Span\, S} = X',\ x'(x_n) \to x'(x)$ for every $x \in X$, show that $x_n \xrightarrow{w} x$ in X (An, CO 1)

5. Let $A \in BL(H)$. Show that the zero space of A is the orthogonal complement of the range of A^*. Hence show that A is one-one if and only if $R(A^*)$ is dense in H. (An, CO 4)

6. Let $A, B \in BL(H)$, where H is a Hilbert space. Show that $(AB)^* = B^*A^*$. (A, CO 3)

7. Let $\{u_n : n = 1, 2, \ldots\}$ be an orthonormal set in a Hilbert space H and let (k_n) be a sequence of scalars. If there exists $x \in H$ such that for $n = 1, 2, \ldots$ such that $< x, u_n >= k_n$, show that $\sum\limits_{n=1}^{\infty} |k_n|^2 < \infty$. (An, CO 2)

8. Show that every Hilbert space is a Banach space. Is the converse true? Justify your answer. (An, CO 2)

9. Let $A \in BL(H)$ be self adjoint. If $< A(x), x >= 0$ for all $x \in H$, show that $A = 0$. (An, CO 4)

10. Define an inner product space. (R, CO 2)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Let H be a Hilbert space over K. For $f \in H'$, let $T(f)$ be the representer of f. For $y \in H$, let $j_y : H' \to K$ be defined by $j_y(f) = f(y)$ for all $f \in H'$. Show that $j_y \in H''$ and the map $J : H \to H''$ defined by $J(y) = j_y$ is linear, onto and satisfies $||J(y)|| = ||y||$ for all $y \in H$. (E, CO 3)

12. Let $P \in BL(H)$. If P is an orthogonal projection, show that $P \geq 0$. (An, CO 4)

13. Let $(X_1, <,>), (X_2, <,>), \ldots$ be a sequence of $ips's$.Let $X = \{x = (x_1, x_2, \ldots) : x_j \in X_j, j = 1, 2, \ldots$ and $\sum\limits_{j=1}^{\infty} < x_j, x_j >_j < \infty\}$. For $x, y \in X$ define $< x, y >= \sum\limits_{j=1}^{\infty} < x_j, y_j >_j$. Show that $<,>$ is an inner product on X. If each X_j is Hilbert, prove that X also is Hilbert. (A, CO 2)

14. Let X be an ips. Show that the adjoint of $A \in BL(X)$ need not exist if X is not complete. (An, CO 3)

15. Let X and Y be $nls's$ and $F : X \to Y$ be linear. If F is compact, show that it maps every weak convergent sequence in X to a convergent sequence in Y. Show that the converse holds if X is reflexive. Show that the condition of reflexivity cannot be omitted in the converse. (E, CO 1)

16. Let X be a Banach space. If every bounded sequence in X has a weak convergent subsequence, show that every non-empty closed subset of X contains an element of minimal norm. (E, CO 1)

17. State and prove the Schwarz's inequality in an ips X. (An, CO 2)

18. Let $A \in BL(H)$. Show that A is normal if and only if $||A(x)|| = ||A^*(x)||$, for all $x \in X$ (An, CO 4)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. (a) Define weak convergence in a normed linear space X. If $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$, prove that $x = y$.
(b) If $x_n \xrightarrow{w} x,\ y_n \xrightarrow{w} y$ in a normed linear space X over K, and $k_n \longrightarrow k$ in K, show that $x_n + y_n \xrightarrow{w} x + y$ and $k_n x_n \xrightarrow{w} kx$.
(c) Show that if $x_n \longrightarrow x$ in a normed linear space X over K, then $x_n \xrightarrow{w} x$ in X. Is the converse true? Justify your answer.
(d) If $x_n \xrightarrow{w} x$ in a normed linear space X over K, show that $x \in \overline{co}\{x_1, x_2, \ldots\}$. (An, CO 1)

20. Let P and Q be orthogonal projections. Show that the following statements are equivalent.
(a) $R(Q) \subseteq Z(P) = R(P)^{\perp}$
(b) PQ=0
(c) P+Q is an orthogonal projection.
Further if $P + Q$ is an orthogonal projection, show that $R(P + Q)$ is the closure of the linear span of $R(P)$ and $R(Q)$. (E, CO 4)

21. Let H be a non-zero Hilbert space. Show that the following are equivalent.
(a) H admits a countable orthonormal basis.
(b) H is linearly isometric with either K^n for some $n = 1, 2, \ldots$, or with l^2 in the $||\ ||_2$ norm.
(c) H is separable. (E, CO 2)

22. Show that every bounded sequence in a Hilbert space H contains a weakly convergent subsequence. (An, CO 3)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Define different types of convergence of a sequence in a Normed space, Inner product space and Hilbert space ,spectral theory of different types of operators ,to relate weak and strong convergence	E	2, 4, 15, 16, 19	11
CO 2	Explain parallelogram law and its geometrical interpretation, inner product and its geometrical application, Schwarz‘s inequality, Pythagoras theorem and its application in geometry,Bessel inequality, projection theorem and Riesz representation theorem.	E	7, 8, 10, 13, 17, 21	12
CO 3	Solve problems based on inner product space and Hilbert space, problems related to strong and weak convergence. To solve problems on spectral theory of different types of operators .To apply spectral theory in solving operator equations.	E	1, 3, 6, 11, 14, 22	12
CO 4	analyze the role of Spectral theory in the study of differential equations and integral equations examine how Functional analysis is closely associated with applied papers like theory of wavelets , signal analysis etc	E	5, 9, 12, 18, 20	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;