Reg. No .... Name **26P404**

M. Sc. DEGREE END SEMESTER EXAMINATION - MARCH 2026

SEMESTER 4 : MATHEMATICS

COURSE : **24P4MATT16 : FUNCTIONAL ANALYSIS - II**

(For Regular 2024 Admission)

Time : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Determine the adjoint of a diagonal operator on a Hilbert space H, which is represented by the diagonal matrix $dia(k_1, k_2, \ldots)$. (A)
2. For $A, B \in BL(H)$, where H is a Hilbert space, show that $||AB|| \leq ||A||||B||$. (A)
3. Define orthonormal basis of a Hilbert space. Give examples of orthonormal bases for K^n, l^2 and $L^2[-\pi, \pi]$. (U)
4. Let $A \in BL(H)$ be self adjoint. If $< A(x), x >= 0$ for all $x \in H$, show that $A = 0$. (An)
5. Show that convergence and weak convergence coincide in a finite dimensional normed linear space. (E)
6. If E is an orthonormal subset of an $ips\ X$, show that $||u - v|| = \sqrt{2}$ for any $u \neq v \in E$. (A)
7. Let $A, B \in BL(H)$, where H is a Hilbert space. Show that $(A + B)^* = A^* + B^*$. (A)
8. Let $\{u_\alpha\}$ be an orthonormal basis for H. If $A \in BL(H)$ is unitary, show that $\{A(u_\alpha\ )\}$ is also an orthonormal basis of H. (U)
9. Suppose that $\{A_n\}$ is a sequence of bounded linear operators on H such that $A_n \to A$ in $BL(H)$. If A_n is self adjoint for $n = 1, 2, \ldots$, show that A is self adjoint. (An)
10. State the Bessel's inequality. (R)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Let H be a Hilbert space. If $A \in BL(H)$, show that there is a unique $B \in BL(H)$ such that $< A(x), y >=< x, B(y) >$, for all $x, y \in H$. (An)
12. Let H be a non-zero Hilbert space. Show that if H is separable, then H admits a countable orthonormal basis. (E)
13. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H.If $\{u_\alpha\}$ is an orthonormal basis for H, show that for $x \in H$, $||x||^2 = \sum\limits_{n=1}^{\infty} | < x, u_n > |^2$ where $\{u_\alpha :< x, u_\alpha >\neq 0\} = \{u_n : n = 1, 2, \ldots\}$. (An)
14. Let X be a normed linear space and assume that X' is separable. Show that a bounded sequence in X need not contain a weak convergent subsequence. (E)

15. Let $A \in BL(H)$. Show that A is unitary if and only if $||A(x)|| = ||x||, \forall x \in X$ and A is onto. (An)

16. Let H be a Hilbert space over K of finite dimension n. If $K = \mathbb{C}$, show that the spectrum of every operator on H consists of n eigen values counting multiplicities. (E)

17. Let f be a continuous linear functional on a Hilbert space H. If $y \in H$ is the representer of f and $\{u_\alpha\}$ is an orthonormal basis for H, show that $y = \sum_{n=1}^{\infty} \overline{f(u_n)} u_n$, where $\{u_n : n = 1, 2, \ldots\} = \{u_\alpha : f(u_\alpha) \neq 0\}$. (An)

18. Let X be a separable normed linear space. Show that every bounded sequence in X' contains a weak* convergent subsequence. Can the condition of separability of X be dropped? Justify your answer. (E)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Let $<, >$ be an inner product on a linear space X. For $x \in X$, let $||x||$ denote the non-negative square root of $< x, x >$.
(a) Show that $| < x, y > | \leq ||x|| \, ||y||$ for every $x, y \in X$, where equality holds if and only if x and y are linearly independent.
(b) Show that $|| \, || : X \to K$ is a norm on X and $<, >: X \times X \to K$ is a continuous function.
(c) Show that the nls $\left(X, || \, ||\right)$ is uniformly convex. (An)

20. (a) Define weak convergence in a normed linear space X. If $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$, prove that $x = y$.
(b) If $x_n \xrightarrow{w} x, \ y_n \xrightarrow{w} y$ in a normed linear space X over K, and $k_n \longrightarrow k$ in K, show that $x_n + y_n \xrightarrow{w} x + y$ and $k_n x_n \xrightarrow{w} kx$.
(c) Show that if $x_n \longrightarrow x$ in a normed linear space X over K, then $x_n \xrightarrow{w} x$ in X. Is the converse true? Justify your answer.
(d) If $x_n \xrightarrow{w} x$ in a normed linear space X over K, show that $x \in \overline{co}\{x_1, x_2, \ldots\}$. (An)

21. Let H be a Hilbert space over K.
(a) For $f \in H'$, let $T(f)$ be the representer of f. Show that the map $T : H' \to H$ is conjugate linear, onto and satisfies $||T(f)|| = ||f||$.
(b) Show that the dual H' of H is a Hilbert space with respect to the inner product defined by $< f, g >' = < T(g), T(f) >$ for all $f, g \in H'$
(c) For $y \in H$, let $j_y : H' \to K$ be defined by $j_y(f) = f(y)$ for all $f \in H'$. Show that $j_y \in H''$ and the map $J : H \to H''$ defined by $J(y) = j_y$ is linear, onto and satisfies $||J(y)|| = ||y||$ for all $y \in H$. (E)

22. State and prove the spectral theorem for compact self-adjoint operators. (E)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;