Reg. No .................... Name .............................. **25P2035**

**M. Sc. DEGREE END SEMESTER EXAMINATION - APRIL 2025**

**SEMESTER 2 : MATHEMATICS**

COURSE : **24P2MATT08 : ALGEBRA -II**

*(For Regular 2024 Admission)*

Time : Three Hours Max. Weights: 30

**PART A**

**Answer any 8 questions** **Weight: 1**

1. Show that $1+\sqrt{2}$ is an algebraic number. (An, CO 2)
2. Do complex zeroes of polynomials with complex coefficients occur in conjugate pairs? Justify your answer (U, CO 3)
3. How many polynomials(including the zero polynomial) are there of degree $\leq 3$ in $\mathbb{Z}_2[x]$? (A, CO 1)
4. Is $\mathbb{Q}[x]\big/\langle x^2+6x+6\rangle$ a field? Justify your answer. (An, CO 1)
5. Find all zeroes of the polynomial $f(x)=x^5+3x^3+x^2+2x$ in $\mathbb{Z}_5$. (A, CO 1)
6. Define primitive $n^{th}$ root of unity in a field. Give an example. (U, CO 3)
7. True or False: $\mathbb{R}$ is not perfect. Justify your answer. (A, CO 4)
8. Show that $\sqrt{\sqrt[3]{2}-i}$ is an algebraic number. (A, CO 2)
9. Find deg$(\sqrt{2},\mathbb{R})$. Is it equal to deg$(\sqrt{2},\mathbb{Q})$? Justify your answer. (An, CO 2)
10. True or False: $\mathbb{C}$ is a splitting field over $\mathbb{R}$. Justify. (A, CO 4)

**(1 x 8 = 8)**

**PART B**

**Answer any 6 questions** **Weights: 2**

11. Let $E$ be an extension field of $F$, where $F$ has $q$ elements. Let $\alpha\in E$ be algebraic over $F$ of degree n. Prove that $F(\alpha)$ has $q^n$ elements. (A, CO 3)
12. For the evaluation homomorphism $\phi_5:\mathbb{Z}_7[x]\to\mathbb{Z}_7$, evaluate $\phi_5(3x^{106}+5x^{99}+2x^{53})$. (An, CO 1)
13. Let $\{\sigma_i|i\in I\}$ be a collection of automorphisms of a field $E$. Show that the set $E_{\sigma_i}$ of all $a\in E$ left fixed by every $\sigma_i$ for $i\in I$ forms a subfield of $E$. What is this subfield usually referred to as? (An, CO 3)
14. Show that if $\alpha,\beta\in\overline{F}$ are both separable over $F$, then $\alpha\pm\beta$, $\alpha\beta$, and $\alpha/\beta$, if $\beta\neq 0$, are all separable over $F$. (An, CO 4)
15. Prove that $x^2-3$ is irreducible over $\mathbb{Q}(\sqrt[3]{2})$. (An, CO 2)
16. What are the conjugates in $\mathbb{C}$ of $\sqrt{3+\sqrt{2}}$ over $\mathbb{Q}$? Justify your answer. (A, CO 3)
17. Show that $x^4-22x+1$ is irreducible over $\mathbb{Q}$. (E, CO 1)
18. Show that all zeroes in $\overline{F}$ of a irreducible polynomial $f(x)\in F[x]$ have the same multiplicity. (E, CO 4)

**(2 x 6 = 12)**

## PART C

**Answer any 2 questions** **Weights: 5**

19. (a). Show that a polynomial $f(x) \in F[x]$ of degree 2 or 3 is reducible over $F$ if and only if it has a zero in $F$. Is the result true for polynomials of degree $\geq 4$? Justify your answer.
(b). How is the reducibility of polynomials over $\mathbb{Z}$ related over their reducibility over $\mathbb{Q}$? Show that if $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{Z}[x]$ with $a_0 \neq 0$ and if $f(x)$ has a zero in $\mathbb{Q}$, then it has a zero $m \in \mathbb{Z}$ and $m$ must divide $a_0$.
(c). Show that $f(x) = x^4 - 2x^2 + 8x + 1$ is irreducible over $\mathbb{Q}$. (E, CO 1)

20. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $F = \mathbb{Q}$. Show that $K$ is a finite separable extension of $F$. (E, CO 4)

21. Stating the necessary lemmas, establish the existence and uniqueness of $\mathbf{GF}(p^n)$, the Galois field of order $p^n$. (E, CO 3)

22. State and prove Kronecker's Theorem. (E, CO 2)

**(5 x 2 = 10)**

OBE: Questions to Course Outcome Mapping

| CO | Course Outcome Description | CL | Questions | Total Wt. |
|---|---|---|---|---|
| CO 1 | Explain ring of polynomials, polynomial factorisation and the ideal structure in F{x} | E | 3, 4, 5, 12, 17, 19 | 12 |
| CO 2 | Comprehend the concept of field extension and the types of extensions. | E | 1, 8, 9, 15, 22 | 10 |
| CO 3 | Analyze finite fields and field automorphisms. | E | 2, 6, 11, 13, 16, 21 | 13 |
| CO 4 | Analyze splitting fields, separable extensions and the main theorem of Galois theory | E | 7, 10, 14, 18, 20 | 11 |

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;