Reg. No .... Name **25P2051**

M. Sc. DEGREE END SEMESTER EXAMINATION - APRIL 2025

SEMESTER 2 : MATHEMATICS

COURSE : **24P2MATT09 : ALGEBRAIC NUMBER THEORY**

(For Regular - 2024 Admission)

Time : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Define cyclotomic field. (R, CO 2)
2. Prove or disprove: $(1+\sqrt{17})/(2-\sqrt{19})$ is an algebraic number but not an algebraic integer. (A, CO 1)
3. If $\mathfrak{a} \neq 0$ is an ideal of $\mathfrak{O}$ with $\mathrm{N}(\mathfrak{a})$ is prime , prove that $\mathfrak{a}|\mathrm{N}(\mathfrak{a})$ (A, CO 4)
4. Prove that the ring of integers $\mathfrak{O}$ in a number field K is noetherian. (U, CO 3)
5. Define Trace of $\alpha \in K$ (U, CO 2)
6. Define a finite extension of a field. (R, CO 1)
7. Let D be a domain and x and y non-zero elements of D. Prove that x and y are associates if and only if $\langle x\rangle = \langle y\rangle$. (A, CO 3)
8. Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of $\mathfrak{O}$. Prove that $\mathfrak{a}|\mathfrak{b}$ iff $\mathfrak{a} \supseteq \mathfrak{b}$ (An, CO 4)
9. Find the group of units of the ring of integers in $\mathbb{Q}\left(\sqrt{-1}\right)$. (U, CO 3)
10. Prove or disprove: K-conjugates of α are distinct. (A, CO 1)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Find the group of units of the ring of integers in $\mathbb{Q}\left(\sqrt{d}\right)$ where $d < 0$. (U, CO 3)
12. Evaluate $N(1-\zeta)$. (A, CO 2)
13. Let θ be a complex number satisfying a monic polynomial equation whose coefficients are algebraic integers. Show that θ is an algebraic integer. (An, CO 1)
14. Prove that $\mathfrak{O}$ of $\mathbb{Q}\left(\sqrt{-5}\right)$ is not a unique factorization domain. (An, CO 3)
15. If $\mathfrak{a}$ is proper ideal of $\mathfrak{O}$, prove that $\mathfrak{a}^{-1} \supsetneqq \mathfrak{O}$ (An, CO 4)
16. Find all monomorphisms from $\mathbb{Q}(\sqrt[3]{7}) \to \mathbb{C}$. (A, CO 1)
17. If $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero ideals of $\mathfrak{O}$, prove that there exists $\alpha \in \mathfrak{a}$ such that $\alpha\mathfrak{a}^{-1} + \mathfrak{b} = \mathfrak{O}$. (A)
18. Find the integral basis and discriminant for $\mathbb{Q}(\sqrt{3})$. (A, CO 2)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Classify the ring of integers of $\mathbb{Q}(\zeta)$. (An, CO 2)
20. Prove that factorization of elements of $\mathfrak{O}$ into irreducibles is unique if and only if every ideal of $\mathfrak{O}$ is principal. (A, CO 4)
21. Prove that a complex number θ is an algebraic integer if and only if the additive group generated by all powers $1, \theta, \theta^2, \cdots$ is finitely generated. (An, CO 1)

22. Prove that in a domain, in which factorization into irreducibles is possible, factorization is unique if and only if every irreducible is prime. (A, CO 3)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Explain the properties of algebraic numbers, algebraic integers and integral bases.	U	2, 6, 10, 13, 16, 21	12
CO 2	Distinguish quadratic and cyclotomic extensions and traces and norms.	A	1, 5, 12, 18, 19	11
CO 3	Analyze factorization into irreducibles in Euclidean Domains and quadratic fields	An	4, 7, 9, 11, 14, 22	12
CO 4	Analyze prime factorization of Ideals, the norm of an ideal, nonunique factorization of cyclotomic fields	An	3, 8, 15, 20	9

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;