Reg. No .... Name **25P4016**

MSc DEGREE END SEMESTER EXAMINATION- MARCH 2025

SEMESTER 4 : MATHEMATICS

COURSE : **21P4MATTEL17 : DIFFERENTIAL GEOMETRY**

(For Regular - 2023 Admission and Supplementary 2022/2021 Admissions)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Sketch the level sets $f^{-1}(c)$ for $n = 0, 1, 2$, where $f(x_1, \ldots, x_{n+1}) = x_{n+1}$ and $c = -1, 0, 1, 2$ (A, CO 1)
2. Sketch the vector field on $\mathbb{R}^2$: $\mathbb{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (0, 1)$ (A, CO 1)
3. Show that parallel vector fields form vector space. (U)
4. Define parametrization of a segment of the plane curve C containing p. (U, CO 3)
5. Define geodesic. (R, CO 2)
6. Define Fermi parallel vector field. (R, CO 2)
7. Describe the graphs and level sets(level curves) of $f(x_1, x_2) = x_1^2 - x_2^2$. (U, CO 1)
8. $L_p(\overline{v}).\,\overline{v}$ represents which property of the surface. (An, CO 4)
9. Let S be an $n-$surface in $\mathbb{R}^{n+1}$ and let $f : S \to \mathbb{R}^k$. Then write the characterization for f to be smooth. (U, CO 4)
10. Find the length of the parametrized curve $\alpha(t) = (t^2, t^3),\ I = [0, 2]$ (A, CO 3)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Write a short note on the parametrized surface of the revolution. (A, CO 4)
12. Compute $\nabla_v f$ where $f(q) = q \cdot q$, $\mathbf{v} = (p, v)$. (A, CO 3)
13. Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field $\mathbb{N}$. Let $p \in S$ and $v \in S_p$. Prove that for every parametrized curve $\alpha : I \to S$, with $\dot{\alpha}(t_0) = v$ for some $t_0 \in I$, $\ddot{\alpha}(t_0) \cdot \mathbb{N}(p) = L_P(v) \cdot v$. (A, CO 3)
14. Find the Gaussian curvature of $\varphi(t, \theta) = (\cos\theta, \sin\theta, t)$. (A, CO 4)
15. Show that the great circles are geodesics in the $2-$sphere. (A, CO 2)
16. Let S denote the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in$\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some $a, b, c, d \in \mathbb{R}$. (A, CO 2)
17. Let U be an open set in $\mathbb{R}^{n+1}$. $\mathbb{X}$ be a smooth vector field on U. Suppose $\alpha : I \to U$ is an integral curve of $\mathbb{X}$ with $\alpha(0) = \alpha(t_0)$ for some $t_0 \in I,\ t_0 \neq 0$. Show that α is periodic. (A, CO 1)
18. Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ (A, CO 1)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$. (A, CO 1)

20. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by
$\eta = -\dfrac{x_2}{x_1^2 + x_2^2} dx_1 + \dfrac{x_1}{x_1^2 + x_2^2} dx_2$. Prove that for $\alpha : [a, b] \to \mathbb{R}^2 - \{0\}$, any closed piecewise smooth parametrized curve in $\mathbb{R}^2 - \{0\}$, $\int_\alpha \eta = 2\pi k$ for some integer k. (A, CO 3)

21. Find the Gaussian curvature of $\varphi(t, \theta) = (t \cos\theta, t \sin\theta, \theta)$. (A, CO 4)

22. State and prove the existence and uniqueness of maximal geodesics. (A, CO 2)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Perceive the ideas of graphs and level sets, vector fields, the tangent space, surfaces, vector fields on surfaces, orientation.	U	1, 2, 7, 17, 18, 19	12
CO 2	Explain the fundamentals of the Gauss map, geodesics, and parallel transport.	A	5, 6, 15, 16, 22	11
CO 3	Summarize the ideas of the Weingarten map, the curvature of plane curves, arc length, and line integrals.	An	4, 10, 12, 13, 20	11
CO 4	Estimate the curvature of surfaces	E	8, 9, 11, 14, 21	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;