Reg. No .... Name **25P4003**

MSc DEGREE END SEMESTER EXAMINATION- MARCH 2025

SEMESTER 4 : MATHEMATICS

COURSE : **21P4MATTEL16 : SPECTRAL THEORY**

(For Regular - 2023 Admission and Supplementary 2022/2021Admissions)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Let A be a compact operator on a Banach space X. Show that every non-zero spectral value of A is an eigen value of A. (A, CO 1)
2. Let f be a continuous linear functional on a Hilbert space H. Define the representer of f. Show that it is unique. (A, CO 3)
3. Let $A \in BL(H)$ and $s(A)$ be the spectrum of A. Show that $k \in S(A)$ if and only if $\bar{k} \in S(A^*)$. (An, CO 4)
4. Let $A \in BL(H)$, where H is a Hilbert space over K. Show that $(A^*)^* = A$ (A, CO 3)
5. Show that every non-zero Hilbert space has an orthonormal basis. (An, CO 2)
6. Let $\{u_\alpha\}$ be an orthonormal basis for H. If $A \in BL(H)$ is unitary, show that $\{A(u_\alpha\})$ is also an orthonormal basis of H. (An, CO 4)
7. If E is an orthogonal subset of non-zero elements of an ips X, show that E is linearly independent. (An, CO 2)
8. Let $X = l^p, 1 \leq p \leq \infty$. Let (k_n) be a sequence in K, and let $A(x(1), x(2), \ldots) = (k_1 x(1), k_2 x(2), \ldots)$. Show that A is compact if and only if $k_n \to 0$. (A, CO 1)
9. For $A, B \in BL(H)$, where H is a Hilbert space, show that $||AB|| \leq ||A|| ||B||$. (A, CO 3)
10. Let $\{u_n : n = 1, 2, \ldots\}$ be an orthonormal set in a Hilbert space H and let (k_n) be a sequence of scalars. If $\sum_{n=1}^{\infty} k_n u_n$ converges in H, show that there exists $x \in H$ such that $< x, u_n >= k_n$ for $n = 1, 2, \ldots$ (An, CO 2)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Let $<,>$ be an inner product on a linear space X. For $x \in X$, let $||x||$ denote the non-negative square root of $< x, x >$. Show that the nls $\left(X, || \, ||\right)$ is uniformly convex. (An, CO 2)
12. Let X be a reflexive normed linear space. Show that every closed subspace of X is reflexive. (An, CO 1)
13. Let A be a compact operator on a Banach space X. Show that $R(A - I)$ is closed in X. (E, CO 1)
14. Let $P \in BL(H)$. If P is normal, show that P is an orthogonal projection. (E, CO 4)
15. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. If for $x \in H,\ x = \sum_{n=1}^{\infty} < x, u_n > u_n$, where $\{u_\alpha :< x, u_\alpha >\neq 0\} = \{u_n : n = 1, 2, \ldots\}$, show that $||x||^2 = \sum_{n=1}^{\infty} | < x, u_n > |^2$ where $\{u_\alpha :< x, u_\alpha >\neq 0\} = \{u_n : n = 1, 2, \ldots\}$ (An, CO 2)

16. Let $H = L^2([a, b])$ and $z \in L^\infty([a, b])$. Define $A : H \to H$ by $A(x) = xz$ for $x \in H$. Show that $A \in BL(H)$ and $||A|| = ||z||_\infty$. (An, CO 3)

17. Let $K = \mathbb{C}$ and $A \in BL(H)$. Show that there are unique self-adjoint operators B and C in $BL(H)$ such that $A = B + iC$. Also show that A is normal if and only if $BC = CB$. Further show that A is unitary if and only if $BC = CB$ and $B^2 + C^2 = I$. (A, CO 4)

18. Let H be a Hilbert space. If $A \in BL(H)$, show that there is a unique $B \in BL(H)$ such that $< A(x), y >=< x, B(y) >$, for all $x, y \in H$. (An, CO 3)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Show that $A \in BL(H)$ is invertible in $BL(H)$ if and only if A is bounded below and the range of A is dense in H. (An, CO 4)

20. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H if and only if for $x \in H$, $||x||^2 = \sum_{n=1}^{\infty} | < x, u_n > |^2$ where $\{u_\alpha :< x, u_\alpha >\neq 0\} = \{u_n : n = 1, 2, \ldots\}$ (E, CO 2)

21. (a) Let X be a separable normed linear space. Show that every bounded sequence in X' contains a weak* convergent subsequence. Can the condition of separability of X be dropped? Justify your answer. (E, CO 1)
(b) Let X be a normed linear space and assume that X' is separable. Show that a bounded sequence in X need not contain a weak convergent subsequence.

22. State and prove the projection theorem. (E, CO 3)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Define different types of convergence of a sequence in a Normed space, Inner product space and Hilbert space ,spectral theory of different types of operators ,to relate weak and strong convergence	E	1, 8, 12, 13, 21	11
CO 2	Explain parallelogram law and its geometrical interpretation, inner product and its geometrical application, Schwarz's inequality, Pythagoras theorem and its application in geometry,Bessel inequality, projection theorem and Riesz representation theorem.	E	5, 7, 10, 11, 15, 20	12
CO 3	Solve problems based on inner product space and Hilbert space, problems related to strong and weak convergence. To solve problems on spectral theory of different types of operators .To apply spectral theory in solving operator equations.	E	2, 4, 9, 16, 18, 22	12
CO 4	analyze the role of Spectral theory in the study of differential equations and integral equations examine how Functional analysis is closely associated with applied papers like theory of wavelets , signal analysis etc	E	3, 6, 14, 17, 19	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;