Reg. No .... Name **24P1019**

M. Sc. DEGREE END SEMESTER EXAMINATION - NOVEMBER 2024

SEMESTER 1 : MATHEMATICS

COURSE : **24P1MATT02 : ALGEBRA - I**

(For Regular - 2024 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Let H be a subgroup of an abelian group G.What is its normalizer $N[H]$ in G? (R, CO 4)
2. Prove that every group of order 35^3 has a normal subgroup of order 125. (A, CO 4)
3. Give an example of an infinite finitely generated abelian group. (U, CO 1)
4. What are the possible numbers of Sylow 3-subgroups of a group of order 255? (A)
5. Find the order of $(3, 10, 9)$ in the group $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$. (A, CO 1)
6. Explain the concept of a free group generated by an alphabet A. (U, CO 3)
7. Is every subnormal series of a group G a normal series of G? Justify your answer. (An, CO 2)
8. Does every abelian group of order divisible by 4 contain a cyclic subgroup of order 4? Justify your answer. (An, CO 1)
9. Give an example of a composition series of a group G which is not a principal series of G (A, CO 1)
10. Is $(3, 0), (0, 1)$ a basis for $\mathbb{Z} \times \mathbb{Z}$? (E, CO 3)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. (a) State the fundamental theorem of Finitely generated abelian groups.
(b) Find all abelian groups upto isomorphism of order 360 (E, CO 1)
12. If N is a normal subgroup of a group G and if H is any subgroup of G, prove that $H \vee N = HN = NH$. Also if H is normal in G, prove that HN is also normal in G. (An, CO 2)
13. Show that a group of order 96 is not simple. (E, CO 4)
14. If G is a nonzero free abelian group with a basis of r elements, show that G is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ for r factors. (An, CO 3)
15. (a) How many subgroups of $\mathbb{Z}_5 \times \mathbb{Z}_6$ are isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_6$? Justify your answer.
(b) How many subgroups of $\mathbb{Z} \times \mathbb{Z}$ are isomorphic to $\mathbb{Z} \times \mathbb{Z}$? Justify your answer. (An, CO 1)
(c) Let p and q be distinct prime numbers. How does the number (up to isomorphism) of abelian groups of order p^r compare with the number (up to isomorphism) of abelian groups of order q^r? Justify your answer.
16. Identify a group isomorphic to the group with presentation $(x, y : y^2x = y, yx^2y = x)$. (E)

17. Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ be the homomorphism such that $\phi(1) = 2$. Find the Kernel K of ϕ. List the cosets in $\mathbb{Z}_{12}/K$ showing the elements in each coset. Give the correspondence between $\mathbb{Z}_{12}/K$ and $\mathbb{Z}_3$ as described in the first isomorphism theorem. (E, CO 2)

18. Prove that every group of order 42 has a normal subgroup of order 7. (E, CO 4)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. (a) If p and q are distinct primes with $p < q$, show that every group of order pq has a unique subgroup of order q and this subgroup is normal in G. Further show that if $q \not\equiv 1 \; (mod\, p)$, then G is abelian and cyclic.
(b) Show that if H and K are finite subgroups of a group G, then
$$|HK| = \frac{(|H|)(|K|)}{|H \cap K|}.$$
(E, CO 1)

20. Show that every group of order 255 is cyclic and abelian. (E, CO 4)

21. (a) State Lagrange's theorem and illustrate it using any suitable example.
(b) Write the converse statement of Lagrange's theorem. Is the converse true for abelian groups? Justify your answer. Is the converse true for non abelian groups? Justify your answer.
(c) State the three Sylow theorems and illustrate each theorem using suitable examples. (E, CO 1)

22. Determine all groups of order 8 up to isomorphism. (E)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Analyze groups using generating sets, direct products, finitely generated abelian groups, and group action on a set.	E	3, 5, 8, 9, 11, 15, 19, 21	18
CO 2	Comprehend Isomorphism Theorems, subnormal/normal series, and solvable groups.	E	7, 12, 17	5
CO 3	Apply free groups and free abelian groups in the proof of the fundamental theorem of abelian groups and in group presentations.	E	6, 10, 14	4
CO 4	Comprehend Sylow theorems and apply the Sylow theory to study groups of different orders.	E	1, 2, 13, 18, 20	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;