Reg. No .... Name **24P2055**

M. Sc. DEGREE END SEMESTER EXAMINATION - MARCH 2024

SEMESTER 2 - MATHEMATICS

COURSE : **21P2MATT10 - MEASURE THEORY AND INTEGRATION**

(For Regular 2023 Admission and Improvement/Supplementary 2022/2021 Admissions)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Let f and g be two measurable functions defined on X, where $(X, \mathcal{B})$ is a measurable space, prove that $f+g$ is a measurable function. (A, CO 4)
2. Prove that the outer measure of a singleton set is zero. Hence prove that the outer measure of a finite set is zero. (A, CO 2)
3. When we say a property holds almost everywhere? If $f \leq g$ a.e., then what is the value of $m\{x : f(x) > g(x)\}$? (U, CO 2)
4. Define the product of the measurable spaces $[[X, \mathcal{S}]]$ and $[[Y, \mathcal{J}]]$. (U, CO 5)
5. Let (X, B, μ) be a measure space and f be a non-negative measurable function defined on X. Prove that the set function ϕ defined as B by $\phi(E) = \int_E f d\mu$ is a measure. (A, CO 4)
6. Show that if f is integrable over E, then so is $|f|$ and $|\int_E f| \leq \int_E |f|$. Does the integrability of $|f|$ imply that of f? Justify. (A, CO 2)
7. Define the characteristic function of a set A. If A and B are two sets, prove that $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B$. (A, CO 2)
8. Define upper and lower Riemann integrals using step functions. (U, CO 2)
9. If μ_1 and μ_2 are two measure on a measurable space (X, B) and a and b are two positive constants, then prove that $a\mu_1 + b\mu_2$ is also a measure. (A, CO 4)
10. Define canonical representation of a simple function. (U, CO 3)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. If $\mathcal{Y}$ in any class of subsets of X, then prove that there exists a smallest monotone class containing $\mathcal{Y}$. (A, CO 5)
12. (a) Prove that χ_A is measurable if and only if A is measurable.
(b) Prove that the set of all points on which a sequence $\langle f_n \rangle$ of measurable functions converges is measurable. (A, CO 2)
13. Let f and g be two non-negative measurable functions such that $f \geq g$. If g is integrable, then show that $\int f - \int g = \int (f-g)$. (A, CO 3)
14. Give an example of a non-measurable set. (A, CO 2)
15. Suppose $(X, \mathcal{B}, \mu)$ is a measure space.
(i) If $E_1, E_2 \in \mathcal{B}$ and $\mu(E_1 \Delta E_2) = 0$, then prove that $\mu E_1 = \mu E_2$
(ii) Show that if μ is complete, $E_1 \in \mathcal{B}$ and $\mu(E_1 \Delta E_2) = 0,$ then $E_2 \in \mathcal{B}$. (A, CO 4)

16. By integrating $e^{-y}\sin 2xy$ w.r.t x and y, show that $\int_0^{\infty} e^{-y}\frac{sin^2 y}{y}dy = \frac{log5}{4}$. (A, CO 5)

17. Prove that every measurable subset of a positive set is positive. Also prove that countable union of positive sets is positive. (An, CO 4)

18. a. If ϕ is a simple function taking the distinct values $a_1, a_2, \ldots, a_n$ on the disjoint measurable sets $A_1, A_2, \ldots, A_n$ respectively, then state the canonical representation of ϕ.
b. If E is any measurable set, prove that $\int_E \phi = \sum_1^n a_i m(A_i \cap E)$. Using it prove that $\int_{A\cup B}\phi = \int_A \phi + \int_B \phi$if A and B are two disjoint measurable sets. (A, CO 3)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. State and prove any two convergence theorems. (A, CO 3)

20. (a) Prove that the collection $\mathcal{M}$ of all measurable sets is a σ-algebra.
(b) Prove that (a, ∞) is measurable for all $a \in R$. (A, CO 1)

21. Prove that $\mathcal{S} \times \mathcal{J} = \mathcal{M}_{\circ}(\mathcal{E})$. (A, CO 5)

22. (a) State and prove Hahn decomposition theorem.
(b) State Jordan Decomposition theorem (A, CO 4)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	define measurable set, measurable function , Lebesgue integral and to relate Lebesgue integral with Riemann integral.	A	20	5
CO 2	explain the relevance of Lebesgue integration	A	2, 3, 6, 7, 8, 12, 14	9
CO 3	solve problems related to Lebegue integral , Lebesgue and abstract measure, Lebesgue and abstract outer measure , signed measure ,Integral with respect to a measure , Integral with respect to product measure .	A	10, 13, 18, 19	10
CO 4	analyse the algebraic properties of Lebesgue integrable functions and Lebesgue measurable functions.	An	1, 5, 9, 15, 17, 22	12
CO 5	develop an algebraic as well as a geometrical structure for the collection of all integrable functions.	Cr	4, 11, 16, 21	10

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;