Reg. No .... Name **24P2042**

M. Sc. DEGREE END SEMESTER EXAMINATION - MARCH 2024

SEMESTER 2 - MATHEMATICS

COURSE : **21P2MATT09 - NUMBER THEORY**

(For Regular 2023 Admission and Improvement/Supplementary 2022/2021 Admissions)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Find the group of units of the ring of integers in $\mathrm{Q}\left(\sqrt{-1}\right)$. (U, CO 4)
2. Prove that factorization into irreducible is possible in O. (A, CO 4)
3. Assume $(a, m) = d$. Prove that the linear congruence $ax \equiv b(\mod m)$ has solutions, if and only if, $d \mid b$. (A, CO 2)
4. Solve the congruence $5x \equiv 3(\mod 24)$. (U, CO 2)
5. True or false : If a non zero ideal a is prime, then $\mathrm{N(a)}$ is a prime. Justify. (A, CO 5)
6. Prove that the m residue classes $\hat{1}, \hat{2}, \ldots, \hat{m}$ are disjoint and their union is the set of all integers. (A, CO 2)
7. Prove that $\sum_{n \le x} \mu(n)\left[\frac{x}{n}\right] = 1$ for $x \ge 1$ (A, CO 1)
8. Prove that Möbius function is not completely multiplicative but multiplicative. (A, CO 1)
9. Suppose that $h = f * g$ and let $H(x) = \sum_{n \le x} h(n),\ F(x) = \sum_{n \le x} f(n)$ and $G(x) = \sum_{n \le x} g(n)$. Prove that $H(x) = \sum_{n \le x} f(n)G(x/n) = \sum_{n \le x} g(n)F(x/n)$. (An, CO 1)
10. If $\mathrm{a} \neq 0$ is an ideal of O with $\mathrm{N(a)}$ is prime , prove that $\mathrm{a} \mid \mathrm{N(a)}$ (A, CO 5)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Derive formula for the divisor sum of Euler totient function. (A, CO 1)
12. Prove that O of $\mathrm{Q}\left(\sqrt{-5}\right)$ is not a unique factorization domain. (An, CO 3)
13. Prove that for $n \ge 1$, $\frac{1}{6} n\log n < p_n < 12\left(n\log n + n\log\left(\frac{12}{e}\right)\right)$ where p_n is the n^{th} prime. (A, CO 2)
14. Let a be an ideal of O. Prove that $\mathrm{N(a)} = \left|\frac{\Delta[\alpha_1, \ldots, \alpha_n]}{\Delta}\right|$, where Δ is the discriminant. (A, CO 4)
15. Prove that the following conditions are equivalent for an integral domain D . (A, CO 3)

1. D is noetherian,
2. D satisfies the ascending chain condition,
3. D satisfies the maximal condition.

16. If a is a non-zero ideal and a$S \subseteq$ a for any set $S \subseteq K$, then prove that $S \subseteq \mathrm{O}$. (A, CO 5)

17. Assume $m_1, \ldots, m_r$ are relatively prime in pairs. Let $b_1, \ldots, b_r$ be arbitrary integers and let $a_1, \ldots, a_r$ satisfy $(a_k, m_k) = 1$ for $k = 1, 2, \ldots, r$. Prove that the linear system of congruences $a_1x \equiv b_1(\mod m_1), \ldots, a_rx \equiv b_r(\mod m_r)$ has exactly one solution modulo $m_1m_2\ldots m_r$. (A, CO 2)

18. Prove that two lattice points (a, b) and (m, n) are mutually visible if, and only if, $a - m$ and $b - n$ are relatievly prime. (An, CO 2)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Prove that every non-zero ideal of O can be written as a product of prime ideals, uniquely up to order of the factors. (An, CO 4)

20. Define Euclidean quadratic Field. Prove that the ring of integers O of $\mathrm{Q}(\sqrt{d})$ is Euclidean for $d = -2, -11$. (U, CO 3)

21. Prove that the following statements are equivalent (A, CO 2)

    1. $\lim_{x \to \infty} \frac{\pi(x)\log x}{x} = 1.$
    2. $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$
    3. $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1.$

22. Prove that if $x \geq 1$ and $\alpha > 0, \alpha \neq 1$ $\sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha + 1)}{\alpha + 1}x^{\alpha+1} + O(x^\beta)$ where $\beta = \max\{1, \alpha\}$. (An, CO 1)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1		U	7, 8, 9, 11, 22	10
CO 2		A	3, 4, 6, 13, 17, 18, 21	14
CO 3		A	12, 15, 20	9
CO 4		An	1, 2, 14, 19	9
CO 5		An	5, 10, 16	4

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;