Reg. No .................... Name .............................. **24U604**

**B. Sc. DEGREE END SEMESTER EXAMINATION - MARCH 2024**

**SEMESTER 6 - MATHEMATICS**

COURSE : **19U6CRMAT09 - REAL ANALYSIS - 2**

*(For Regular 2021 Admission and Supplementary 2020/2019 Admissions)*

Time : Three Hours Max. Marks: 75

**PART A**
**Answer any 10 (2 marks each)**

1. Show that the series $\sum r^n \cos n\theta,\ 0 < r < 1$, converges uniformly for all real values of $\theta$.
2. Evaluate $\int_0^1 x^4(1-x)^2\,dx$.
3. Define a partition of $[a, b]$.
4. Show that the series $\sum r^n \sin n\theta,\ 0 < r < 1$, converges uniformly for all real values of $\theta$.
5. Show that a function which is uniformly continuous on an interval is continuous on that interval.
6. Define the improper integral $\int_a^\infty f\,dx$, where $x \geq a$.
7. Discuss the discontinuity of the function defined by $f(x) = x - [x],\ \forall\, x \geq 0$, at $x = 3$
8. State the Darboux's condition of Riemann integrability of a bounded function $f$ on $[a, b]$.
9. Define pointwise convergence of a sequence of functions $\{f_n\}$.
10. Show that the function defined by $f(x) = \begin{cases} x \sin 1/x & when\, x \neq 0 \\ 0 & when\, x = 0 \end{cases}$ is continuous at $x = 0$.
11. Show that $\Gamma(1) = 1$
12. Prove that for any two partitions $P_1$ and $P_2$ of $[a, b]$ and for a bounded function $f$, $L(P_1, f) \leq U(P_2, f)$.

**(2 x 10 = 20)**

**PART B**
**Answer any 5 (5 marks each)**

13. State and prove the fixed point theorem.
14. Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is not uniformly convergent on any interval containing zero.
15. State and prove the generalised first mean value theorem for integrals.
16. Show that the function $f$ defined by $f(x) = \begin{cases} 0 & when\, x\, is\, rational \\ 1 & when\, x\, is\, irrational \end{cases}$ is not Riemann integrable on any interval.
17. Evaluate $\int_0^1 \dfrac{dx}{\sqrt{1-x}}$.

18. The function $f$ is defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} -x^2 & when\, x \le 0 \\ 5x - 4 & when\, 0 < x \le 1 \\ 4x^2 - 3x & when\, 1 < x < 2 \\ 3x + 4 & when\, x \ge 2 \end{cases}.$$

Examine $f$ for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.

19. Test for convergence of the improper integral $\int_1^2 \frac{x dx}{\sqrt{x-1}}$.

20. Show that the sequence $\{f_n\}$, where $f_n(x) = \tan^{-1} nx,\ x \ge 0$ is uniformly convergent on any interval $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$.

**(5 x 5 = 25)**

## PART C
**Answer any 3 (10 marks each)**

21. Let $\{f_n\}$ be a sequence of functions such that $\lim_{n \to \infty} f_n(x) = f(x),\ x \in [a, b]$ and let $M_n = Sup|f_n(x) - f(x)|$ for $x \in [a, b]$ . Prove that $f_n \to f$ uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.

22. Suppose $f$ is bounded and integrable on $[a, b]$ and $F$ is defined by $F(x) = \int_a^x f(t)dt,\ a \le x \le b$. Show that $F$ is continuous on $[a, b]$. Further show that if $f$ is continuous at a point $c$ of $[a, b]$, then $F$ is differentiable at $c$ and $F'(c) = f(c)$.

23. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}$.

24. Show that a function which is continuous on a closed interval is also uniformly continuous on that interval.

**(10 x 3 = 30)**