Reg. No .... Name **24P4015**

M. Sc. DEGREE END SEMESTER EXAMINATION : MARCH 2024

SEMESTER 4 - MATHEMATICS

COURSE : **21P4MATTEL17 - DIFFERENTIAL GEOMETRY**

(For Regular - 2022 Admission and Supplementary - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Prove that the geodesics have constant speed. (A, CO 2)
2. Define Gauss - Kronecker curvature. (U, CO 4)
3. Give Frenet formula for a plane curve. (A, CO 3)
4. Define the height of a level set. (A, CO 1)
5. Sketch the vector field on $\mathbb{R}^2$: $\mathbb{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (-x_1, -x_2)$. (A, CO 1)
6. Describe the level sets of $f(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3$, for $n = 0, 1, 2.$ (U, CO 1)
7. Let f and g be two smooth functions on the open set $J \subset R^{n+1}$ show that $d(f+g) = df + dg$. (A, CO 3)
8. Find the velocity, the acceleration, and the speed of parametrized curve $\alpha(t) = (t, t^2)$ (A, CO 2)
9. Show that parallel vector fields form vector space. (U)
10. Explain the first and the second fundamental form of a surface S. (An, CO 4)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Distinguish between Euclidean parallelism and Levi-Civita parallelism. (U, CO 2)
12. Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field $\mathbb{N}$. Let $p \in S$ and $v \in S_p$. Prove that for every parametrized curve $\alpha : I \to S$, with $\dot{\alpha}(t_0) = v$ for some $t_0 \in I$, $\ddot{\alpha}(t_0) \cdot \mathbb{N}(p) = L_P(v) \cdot v$. (A, CO 3)
13. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$, let $p \in S$, and let $\{k_1(p), \ldots, k_n(p)\}$ be the principal curvatures of S at p with corresponding orthogonal principal curvature directions $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Prove that the normal curvature $k(\mathbf{v})$ in the direction $\mathbf{v} \in S_p$ is given by $k(\mathbf{v}) = \sum_{i=1}^{n} k_i(p)(\mathbf{v} \cdot \mathbf{v}_i)^2 = \sum_{i=1}^{n} k_i(p) \cos^2 \theta_i$ where $\theta_i = \cos^{-1}(\mathbf{v} \cdot \mathbf{v}_i)$. (A)
14. Determine whether the vector field $\mathbb{X}(x_1, x_2) = (x_1, x_2, 1 + x_1^2, 0)$ where $U = \mathbb{R}^2$ is complete or not. (An, CO 1)
15. Explain the surface of revolution S obtained by rotating the curve C about the x_1-axis where $C = f^{-1}(c)$ for some $f : U \to \mathbb{R};\ \ U \subseteq \mathbb{R}^{n+1}$ with $\nabla f(p) \neq 0,\ \ \forall p \in C$. (A, CO 1)
16. Compute $\nabla_v f$ where $f(q) = q \cdot q$, $\mathbf{v} = (p, v)$. (A, CO 3)
17. Explain parallel transport. (U, CO 2)

18. Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self-adjoint linear transformation on V. Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \to \mathbb{R}$ by $f(v) = L(v) \cdot v$. Suppose f is stationary at $v_0 \in S$. Prove that $L(v_0) = f(v_0)v_0$. (An, CO 4)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Let $S = f^{-1}(c)$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function such that $\nabla f(p) \neq 0$ for all $p \in S$. Suppose $\alpha : \mathbb{R} \to \mathbb{R}^{n+1}$ is a parametrized curve which is nowhere tangent to S. (A)
(i) Show that at each pair of consecutive crossings of S by α the direction of the orientation $\nabla f / \|\nabla f\|$ on S reverses relative to the direction of α.
(ii) Show that S is compact and α goes to ∞ in both directions.

20. Let $\varphi : U \to \mathbb{R}^{n+1}$ be a parametrized n-surface in $\mathbb{R}^{n+1}$ and let $p \in U$. Prove that there exists an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$. (A, CO 4)

21. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\mathbb{X}$ be a smooth tangent vector field on S, and let $p \in S$. Prove that there exists an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that (An, CO 1)
(i) $\alpha(0) = p$
(ii) $\dot{\alpha}(t) = \mathbb{X}(\alpha(t))$ for all $t \in I$
(iii) If $\beta : \tilde{I} \to S$ is any other parametrized curve in S satisfying (i) and (ii), then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

22. Define local parametrization of a plane curve C and prove that the local parametrization is unique upto a reparametrization. (An, CO 3)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Perceive the ideas of graphs and level sets, vector fields, the tangent space, surfaces, vector fields on surfaces, orientation.	U	4, 5, 6, 14, 15, 21	12
CO 2	Explain the fundamentals of the Gauss map, geodesics, and parallel transport.	A	1, 8, 11, 17	6
CO 3	Summarize the ideas of the Weingarten map, the curvature of plane curves, arc length, and line integrals.	An	3, 7, 12, 16, 22	11
CO 4	Estimate the curvature of surfaces	E	2, 10, 18, 20	9

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;