Reg. No .... Name **24P4003**

M. Sc. DEGREE END SEMESTER EXAMINATION - MARCH 2024

SEMESTER 4 - MATHEMATICS

COURSE : **21P4MATTEL16 - SPECTRAL THEORY**

(For Regular - 2022 Admission and Supplementary - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. State the Bessel's inequality. (R, CO 2)
2. Let F be a non-empty closed subspace of a Hilbert space H. Show that $F^{\perp\perp} = F$. (A, CO 3)
3. Let $A \in BL(H)$, where H is a Hilbert space over K. Show that $||A^*A|| = ||A||^2$. (A, CO 3)
4. Let H be a Hilbert space and $A \in BL(H)$. Define the adjoint of A. (R, CO 3)
5. Let $A \in BL(H)$ and $s(A)$ be the spectrum of A. Show that $k \in S(A)$ if and only if $\bar{k} \in S(A^*)$. (An, CO 4)
6. Let $A \in BL(H)$ be self adjoint. If $< A(x), x >= 0$ for all $x \in H$, show that $A = 0$. (An, CO 4)
7. Show that every non-zero Hilbert space has an orthonormal basis. (An, CO 2)
8. Let A be a compact operator on a Banach space X. Show that the spectrum of A is countable and has no limit point except possibly the number 0. (A, CO 1)
9. Define weak convergence in a normed linear space X. If $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$, prove that $x = y$. (A, CO 1)
10. Let $\{u_n : n = 1, 2, \ldots\}$ be an orthonormal set in a Hilbert space H and let (k_n) be a sequence of scalars. If $\sum_{n=1}^{\infty} k_n u_n$ converges in H, show that there exists $x \in H$ such that $< x, u_n >= k_n$ for $n = 1, 2, \ldots$ (An, CO 2)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Let $P \in BL(H)$. If P is an orthogonal projection, show that $P \geq 0$. (An, CO 4)
12. Show that $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin n\pi t}{\sqrt{\pi}}, \frac{\cos n\pi t}{\sqrt{\pi}}, n = 1, 2, \ldots \right\}$ is an orthonormal basis for $L^2([-\pi, \pi])$. (E, CO 2)
13. Let X be an ips. Show that the adjoint of $A \in BL(X)$ need not exist if X is not complete. (An, CO 3)
14. If H is a finite dimensional Hilbert space and $A \in BL(H)$ satisfies $AA^* = I$, prove that A is unitary. Does the result hold if H is infinite dimensional?Justify your answer. (A, CO 4)
15. Explain the Gram-Schmidt orthonormalization process. (An)
16. Let A be a compact operator on a Banach space X. Show that $Z(A - I)$ is finite dimensional. (E)
17. Let $H = L^2([a, b])$ and $z \in L^{\infty}([a, b])$. Define $A : H \to H$ by $A(x) = xz$ for $x \in H$. Show that $A \in BL(H)$ and $||A|| = ||z||_{\infty}$. (An, CO 3)

18. Let X and Y be $nls's$ and $F : X \to Y$ be linear. If F is compact, show that it maps every weak convergent sequence in X to a convergent sequence in Y. Show that the converse holds if X is reflexive. Show that the condition of reflexivity cannot be omitted in the converse. (E, CO 1)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Let $K = \mathbb{C}$ and A be a normal operator. Prove that $||A|| = R_A = r_A$. Also show that there is an approximate eigen value k such that $|k| = ||A||$. (E, CO 4)

20. Let H be a Hilbert space over K.
(a) For $f \in H'$, let $T(f)$ be the representer of f. Show that the map $T : H' \to H$ is conjugate linear, onto and satisfies $||T(f)|| = ||f||$.
(b) Show that the dual H' of H is a Hilbert space with respect to the inner product defined by $< f, g >' = < T(g), T(f) >$ for all $f, g \in H'$ (E, CO 3)
(c) For $y \in H$, let $j_y : H' \to K$ be defined by $j_y(f) = f(y)$ for all $f \in H'$. Show that $j_y \in H''$ and the map $J : H \to H''$ defined by $J(y) = j_y$ is linear, onto and satisfies $||J(y)|| = ||y||$ for all $y \in H$.

21. Let A be a compact operator on a Banach space X. Show that $A - I$ is one-to-one if and only if $A - I$ is onto. (E, CO 1)

22. Let $<, >$ be an inner product on a linear space X. For $x \in X$, let $||x||$ denote the non-negative square root of $< x, x >$.
(a) Show that $| < x, y > | \leq ||x|| \, ||y||$ for every $x, y \in X$, where equality holds if and only if x and y are linearly independent. (An, CO 2)
(b) Show that $|| \, || : X \to K$ is a norm on X and $<, >: X \times X \to K$ is a continuous function.
(c) Show that the nls $\left(X, || \, ||\right)$ is uniformly convex.

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Define different types of convergence of a sequence in a Normed space, Inner product space and Hilbert space ,spectral theory of different types of operators ,to relate weak and strong convergence	E	8, 9, 18, 21	9
CO 2	Explain parallelogram law and its geometrical interpretation, inner product and its geometrical application, Schwarz's inequality, Pythagoras theorem and its application in geometry,Bessel inequality, projection theorem and Riesz representation theorem.	E	1, 7, 10, 12, 22	10
CO 3	Solve problems based on inner product space and Hilbert space, problems related to strong and weak convergence. To solve problems on spectral theory of different types of operators .To apply spectral theory in solving operator equations.	E	2, 3, 4, 13, 17, 20	12
CO 4	analyze the role of Spectral theory in the study of differential equations and integral equations examine how Functional analysis is closely associated with applied papers like theory of wavelets , signal analysis etc	E	5, 6, 11, 14, 19	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;