Reg. No .... Name **23P4045**

M. Sc. DEGREE END SEMESTER EXAMINATION : MARCH 2023

SEMESTER 4 : MATHEMATICS

COURSE : **21P4MATTEL20 : THEORY OF WAVELETS**

(For Regular - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Which of the following sequences is square summable?
(i) $(Z(n))_{n=1}^{\infty}$, where $z(n) = \frac{1}{\sqrt{n}}$
(ii)$(w(n))_{n=1}^{\infty}$, where $w(n) = \frac{1}{n}$ (An, CO 3)
2. If $B = \{R_{2k}v\}_{k=0}^{M-1} \bigcup \{R_{2k}u\}_{k=0}^{M-1}$ is a first stage wavelet basis for $l^2(Z_N)$, then represent the construction of $[z]_B$ for any $z \in l^2(Z_N)$ by a filter bank diagram. (U, CO 1)
3. Define the orthogonal direct sum of two subspaces U and V of an innerproduct space X. (U, CO 2)
4. Define the discrete Fourier transform $\wedge : l^2(Z_N) \to l^2(Z_N)$. (An, CO 1)
5. Suppose N is divisible by 2^p. Suppose $u_l, v_l \in l^2(Z(\frac{N}{2}^{l-1})$ for $l = 1, 2, .., p$.
Define $f_1 = v_1, g_1 = u_1$ and for $l = 2, 3, .., p$ define $f_l = g_{l-1} * U^{l-1}(v_l)$, prove that $f_l = u_1 * U(u_2) * U^2(u_3) * ... * U^{l-2}(u_{l-1}) * U^{l-1}(v_l)$. (E, CO 2)
6. Define a complete orthonormal set in a Hilbert space. (U, CO 3)
7. Define a homogeneous wavelet system for $l^2(Z)$. (An, CO 4)
8. Suppose $z \in l^2(Z_N)$. Prove that $\hat{z}$ is real if and only if $z(m) = \overline{z(N-m)}$ for all m. (U, CO 1)
9. Let $\{a_k\}_{k \in Z}$ and $\{b_k\}_{k \in Z}$ be orthonormal sets in a Hilbert space H. with $< a_j, b_k >= 0$ for $j, k \in Z$. Define $V = [\sum_{k \in Z} z(k)a_k : z = (z(k))_{k \in Z} \in l^2(Z)]$ and $W = [\sum_{k \in Z} z(k)b_k : z = (z(k))_{k \in Z} \in l^2(Z)]$. Then prove that $V \perp W$. (An, CO 4)
10. If $\sum_{n \in Z} w(n)$ converges absolutely, prove that $\sum_{n=0}^{\infty} w(n)$ and $\sum_{n=1}^{\infty} w(-n)$ converges absolutely. (A, CO 3)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Suppose N is divisible by 2^p.Suppose u,v $\in l^2(Z_N)$ are such that the system matrix A(n) of u and v is unitary for all n.
Define $u_1 = u$ and $v_1 = v$ and for $l = 2, 3, .., p$ define
$u_l(n) = \sum_{k=0}^{2^{l-1}-1} u_1(n + \frac{kN}{2^{l-1}})$ and $v_l(n) = \sum_{k=0}^{2^{l-1}-1} v_1(n + \frac{kN}{2^{l-1}})$. Then prove that $u_1, v_1, u_2, v_2, \dots u_p, v_p$ is a p^{th} stage wavelet filter sequence. (An, CO 2)
12. Suppose $M \in N, N = 2M, z \in l^2(Z_N)$ and $w \in l^2(Z_M)$. Prove that $< D(z), w >=< z, U(w) >$ (U, CO 1)
13. Let $\hat{u} = (\sqrt{2}, 1, 0, 1)$ and $\check{v} = (0, 1, \sqrt{2}, -1)$
(a) Find u and v
(b) Construct an orthonormal basis for $l^2(Z_4)$ using u and v (A, CO 1)
14. Let $z = (2, 5, -1, i) \in l^2(Z_4)$
(a) Find U(z)
(b) Find D(z) (An, CO 1)

(c) Prove that $UoD(z) = \frac{1}{2}(z + z^*)$
(d) Prove that $DoU(z) = z$

15. Prove that $L^2[(-\pi, \pi)]$ is a normed space. (A, CO 3)

16. Derive a complete orthonormal set in $l^2(Z)$. (A, CO 3)

17. i) Define delta function $'\delta'$.
(ii)Suppose $b \in l^1(Z)$ and define $T_b(z) = b * z$ for all $z \in l^2(Z)$. Then prove that $T_b : l^2(Z) \to l^2(Z)$ is a translation invariant linear transformation. (An, CO 4)

18. Suppose N is divisible by 2^l, x, y, $w \in l^2(Z^l_{N/2})$ and $z \in l^2(Z_N)$. Then prove that $D^l(z) * w = D^l(z * U^l(w))$ and $U^l(x * y) = U^l(x) * U^l(y)$. (E, CO 2)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Describe Daubechie's D_6 wavelet system on Z_N. (E, CO 2)

20. (i) For $k \in Z$, define the translation operator $R_k : l^2(Z) \to l^2(Z)$.
(ii) When we say a linear transformation $T : l^2(Z) \to l^2(Z)$ is translation invariant?.
(iii) Suppose $T : l^2(Z) \to l^2(Z)$ is a bounded translation invariant linear transformation. If we define $b \in l^2(Z)$ by $b = T(\delta)$, then prove that $T(z) = b * z$ for all $z \in l^2(Z)$. (An, CO 4)

21. i) What are the elements of $L^2([-\pi, \pi))$?
ii) Define addition and scalar multiplication in $L^2([-\pi, \pi))$.
iii) Define inner product in $L^2([-\pi, \pi))$ and state the norm induced by the innerproduct.
iv) Using Cauchy-schwarz inequality in an innerproduct space and triangle inequality in a normed space deduce the following
$\int_{-\pi}^{\pi} |f(\theta)g(\theta)| d\theta \leq (\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta)^{1/2} (\int_{-\pi}^{\pi} |g(\theta)|^2 d\theta)^{1/2}$ and
$(\int_{-\pi}^{\pi} |f(\theta) + g(\theta)|^2 d\theta)^{1/2} \leq (\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta)^{1/2} + (\int_{-\pi}^{\pi} |g(\theta)|^2 d\theta)^{1/2}$ for any $f, g \in L^2([-\pi, \pi))$ (U, CO 3)

22. (a) Let $w \in l^2(Z_N)$. Then prove that $\{R_k w\}_{k=0}^{N-1}$ is orthonormal basis for $l^2(Z_N)$ if and if $|\hat{w}(n)| = 1$ for all $n \in Z_N$
(b) If $B = \{R_k w\}_{k=0}^{N-1}$ is an orthonomal basis for $l^2(Z_N)$. Prove that $[z]_B = z * \tilde{w}$. (U, CO 1)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Define first stage and pth stage wavelet basis for l 2[ZN], Fourier transform including discrete case ,complete orthonormal system , first stage wavelet system and homogeneous wavelet system for l2[Z]	U	2, 4, 8, 12, 13, 14, 22	14
CO 2	Explain the filter bank diagram and its use in the construction of the output of the analysis phase of the filter bank	U	3, 5, 11, 18, 19	11
CO 3	Apply theory of wavelets in the frequency analysis of a video or audio signal.	U	1, 6, 10, 15, 16, 21	12
CO 4	Develop wavelet bases for l 2 [ZN] and l2 [Z],both first stage and pth stage	U	7, 9, 17, 20	9

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;