Reg. No .... Name **23P4015**

M. Sc. DEGREE END SEMESTER EXAMINATION : MARCH 2023

SEMESTER 4 : MATHEMATICS

COURSE : **21P4MATTEL17 : DIFFERENTIAL GEOMETRY**

(For Regular - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Find the integral curve through $p = (x_1, x_2) = (1, 1)$ of the vector field $\mathbb{X}(p) = (p, 0, 1)$. (A, CO 1)
2. Define a quadratic form associated with a linear operator L on a finite dimensional vector space with a dot product. (U, CO 4)
3. Define covariant derivative of a parallel vector field. (U, CO 2)
4. Find the velocity, the acceleration, and the speed of parametrized curve $\alpha(t) = (\cos t, \sin t, 2\cos t, 2\sin t)$. (A, CO 2)
5. Sketch the vector field on $\mathbb{R}^2$: $\mathbb{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (x_2, x_1)$. (A, CO 1)
6. Let $\alpha : I \to \mathbb{R}^2$ defined by $\alpha(t) = (\gamma\cos t, \gamma\sin t)$. Find $l(\alpha)$. (A, CO 3)
7. Define the divergence of a smooth vector field. (U, CO 1)
8. Find the length of the given parametrized curve $\alpha : I \to R^3$ defined by $\alpha(t) = \left(\sqrt{2}\cos 2t, \sin 2t, \sin 2t\right), I = [0, 2\pi]$. (A, CO 3)
9. Find the normal curvature of the hyperboloid $-x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$, oriented by the unit normal vector field $\mathbb{N}(p) = (p, -x_1/\|p\|, x_2/\|p\|, x_3/\|p\|)$ at $p = (0, 0, 1)$ in the direction $\mathbf{v} = (1, 0, 0)$. (An, CO 4)
10. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$. (A, CO 2)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. If v_0 is an eigenvector of a self adjoint operator on a finite dimensional vector space V . Prove that $f(v) = L(v) \cdot v$ is stationary at v_0. (An, CO 4)
12. Let S denote the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in$\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some $a, b, c, d \in \mathbb{R}$. (A, CO 2)
13. Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$, and let $\mathbb{X}$ be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of X such that $\alpha(t_0) \in S$ for some $t_0 \in I$, then prove that $\alpha(t) \in S$ for all $t \in I$. (A, CO 1)

14. Sketch the level sets $f^{-1}(c)$ for $n = 0, 1, 2$, where $f(x_1, \ldots, x_{n+1}) = x_1^2 + x_2^2/4 + \cdots + x_{n+1}^2/(n+1)^2$ and $c = 1$ (A, CO 1)

15. Prove that the local parametrization is unique upto a reparameterization. (A)

16. Compute $\nabla_v f$ where $f(x_1, x_2) = x_1^2 - x_2^2$, $v = (1, 1, \cos\theta, \sin\theta)$. (A, CO 3)

17. Find global parametrizations the plane curve $f^{-1}(c)$, oriented by $\nabla f / \|\nabla f\|$ where $f(x_1, x_2) = ax_1 + bx_2, \quad (a, b) \neq (0, 0)$. (A, CO 3)

18. Let S be an n-surface in $\mathbb{R}^{n+1}$ and $p \in S$. Show that the subset of S consisting of all points $q \in S$ which can be joined to p by a continuous curve in S is a connected n-surface. (An, CO 4)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$. (A, CO 1)

20. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p \in S$, and let $v \in S_p$. Prove that there exists an open interval I containing 0 and a geodesic $\alpha : I \to S$ such that
(i) $\alpha(0) = p$ and $\dot{\alpha}(0) = v$. (An, CO 2)
(ii) If $\beta : \tilde{I} \to S$ is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) = v$, then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

21. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by $\eta = -\dfrac{x_2}{x_1^2 + x_2^2} dx_1 + \dfrac{x_1}{x_1^2 + x_2^2} dx_2$. Prove that for $\alpha : [a, b] \to \mathbb{R}^2 - \{0\}$, any closed piecewise smooth parametrized curve in $\mathbb{R}^2 - \{0\}$, $\int_\alpha \eta = 2\pi k$ for some integer k. (A, CO 3)

22. (i) Find the Gaussian curvature of $\phi(t, \theta) = (\cos\theta, \sin\theta, t)$
(ii) Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite. (A, CO 4)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Perceive the ideas of graphs and level sets, vector fields, the tangent space, surfaces, vector fields on surfaces, orientation.	U	1, 5, 7, 13, 14, 19	12
CO 2	Explain the fundamentals of the Gauss map, geodesics, and parallel transport.	A	3, 4, 10, 12, 20	10
CO 3	Summarize the ideas of the Weingarten map, the curvature of plane curves, arc length, and line integrals.	An	6, 8, 16, 17, 21	11
CO 4	Estimate the curvature of surfaces	E	2, 9, 11, 18, 22	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;