Reg. No .... Name **23P4003**

M. Sc. DEGREE END SEMESTER EXAMINATION : MARCH 2023

SEMESTER 4 : MATHEMATICS

COURSE : **21P4MATTEL16 : SPECTRAL THEORY**

(For Regular - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Let $A \in BL(H)$, where H is a Hilbert space over K. Show that $||A^*|| = ||A||$. (A, CO 3)
2. Define orthonormal basis of a Hilbert space. Give examples of orthonormal bases for K^n, l^2 and $L^2[-\pi, \pi]$. (U, CO 2)
3. If X is a finite dimensional nls, show that X is reflexive. (A, CO 1)
4. Let f be a continuous linear functional on a Hilbert space H. Define the representer of f. Show that it is unique. (A, CO 3)
5. Let $A \in BL(H)$. Show that the zero space of A is the orthogonal complement of the range of A^*. Hence show that A is one-one if and only if $R(A^*)$ is dense in H. (An, CO 4)
6. Let $\{u_\alpha\}$ be an orthonormal basis for H. If $A \in BL(H)$ is unitary, show that $\{A(u_\alpha\})$ is also an orthonormal basis of H. (An, CO 4)
7. If E is an orthogonal subset of non-zero elements of an ips X, show that E is linearly independent. (An, CO 2)
8. If $x_n \xrightarrow{w} x$ in a normed linear space X over K, show that $x \in \overline{co}\{x_1, x_2, \ldots\}$. (An, CO 1)
9. Let f be a continuous linear functional on a Hilbert space H. Define the representer of f. Show that it is unique. (A, CO 3)
10. Define a Hilbert space. Show that $L^p[0, 1]$ with p-norm is a Hilbert space if and only if $p = 2$. (A, CO 2)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Let X be a Banach space. If X is reflexive, show that every bounded sequence in X has a weak convergent subsequence. (An, CO 1)
12. Show that the projection theorem need not hold in an inner product space which is not complete. (An, CO 3)
13. Let H be a separable Hilbert space with the ordered orthonormal basis $\{u_1, u_2, \ldots\}$. Let (k_n) be a bounded sequence in K, and $A(x) = \sum\limits_{n=1}^{\infty} k_n < x, u_n > u_n$, for $x \in H$, be a diagonal operator on H. Show that the eigen spectrum of A, $e(A) = \{k_n : n = 1, 2, \ldots\}$. (An, CO 4)
14. Let H be a Hilbert space over K. For $f \in H'$, let $T(f)$ be the representer of f. Show that the dual H' of H is a Hilbert space with respect to the inner product defined by $< f, g >' = < T(g), T(f) >$ for all $f, g \in H'$ (An, CO 3)

15. State and prove Bessel's inequality. (An, CO 2)

16. Let X and Y be $nls's$ and $F : X \to Y$ be linear. If $F \in BL(X, Y)$ and has finite rank, show that $R(F)$ is closed in Y and F is compact. Show that the converse holds if X and Y are Banach. (E, CO 1)

17. Let $\{u_n : n = 1, 2, \ldots\}$ be an orthonormal set in a Hilbert space H and let (k_n) be a sequence of scalars. Show that, there exists $x \in H$ such that $< x, u_n >= k_n$ for $n = 1, 2, \ldots$ if and only if $\sum_{n=1}^{\infty} |k_n|^2 < \infty$. (E, CO 2)

18. Let $A \in BL(H)$. Show that A is unitary if and only if $||A(x)|| = ||x||, \forall x \in X$ and A is onto. (An, CO 4)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. Show that every bounded sequence in a Hilbert space H contains a weakly convergent subsequence. (An, CO 3)

20. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that the following are equivalent.
(i) $\{u_\alpha\}$ is an orthonormal basis for H.
(ii) For $x \in H,\ x = \sum_{n=1}^{\infty} < x, u_n > u_n$, where $\{u_\alpha :< x, u_\alpha >\neq 0\} = \{u_n : n = 1, 2, \ldots\}$
(iii) For $x \in H$, $||x||^2 = \sum_{n=1}^{\infty} | < x, u_n > |^2$ where $\{u_\alpha :< x, u_\alpha >\neq 0\} = \{u_n : n = 1, 2, \ldots\}$
(iv) If $x \in H$ and $< x, u_\alpha >= 0$ for all α, then $x = 0$. (E, CO 2)

21. Let A be a compact operator on a Banach space X. Show that $Z(A - I)$ is finite dimensional and $R(A - I)$ is closed in X. (E, CO 1)

22. Let P and Q be orthogonal projections. Show that the following statements are equivalent.
(a) $R(Q) \subseteq R(P) = Z(P)^{\perp}$
(b) PQ=Q
(c) P-Q is an orthogonal projection
(d) $Q \leq P$
Further if $P - Q$ is an orthogonal projection, prove that $R(P - Q) = R(P) \cap Z(Q)$ (An, CO 4)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Define different types of convergence of a sequence in a Normed space, Inner product space and Hilbert space ,spectral theory of different types of operators ,to relate weak and strong convergence	E	3, 8, 11, 16, 21	11
CO 2	Explain parallelogram law and its geometrical interpretation, inner product and its geometrical application, Schwarz's	E	2, 7, 10, 15, 17, 20	12

	inequality, Pythagoras theorem and its application in geometry,Bessel inequality, projection theorem and Riesz representation theorem.			
CO 3	Solve problems based on inner product space and Hilbert space, problems related to strong and weak convergence. To solve problems on spectral theory of different types of operators .To apply spectral theory in solving operator equations.	E	1, 4, 9, 12, 14, 19	14
CO 4	analyze the role of Spectral theory in the study of differential equations and integral equations examine how Functional analysis is closely associated with applied papers like theory of wavelets , signal analysis etc	E	5, 6, 13, 18, 22	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;