Reg. No .... Name .... **22P315 - S**

M. Sc DEGREE END SEMESTER EXAMINATION - OCTOBER 2022

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT12 : ADVANCED FUNCTIONAL ANALYSIS**

(For Supplementary - 2016/2017/2018/2019/2020 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer All (1.5 marks each)

1. Let (f_n) be bounded linear functional on a Banach space X is $weak^*$ convergent. Prove that the sequence $(||f_n||)$ is bounded.
2. Show that $T : R^2 \to R$ defined by $(\xi_1, \xi_2) \to (\xi_1)$ is open. Is the mapping $T : R^2 \to R^2$ defined by $(\xi_1, \xi_2) \to (\xi_1, 0)$ an open mapping?. Justify.
3. Prove that boundedness does not imply closedness of a linear operator.
4. Define eigen values of a square matrix A. Also find the eigen values of

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

5. Let A be a complex Banach algebra with identity `e' and $x \in A$. Then prove that $r_\sigma(x) \leq ||x||$.
6. Let $T : X \to Y$ be a linear operator where X and Y are normed spaces. If X is finite dimensional then prove that T is compact.
7. Prove that the composite of two linear operators is compact if one is compact and the other is bounded.
8. Define a positive operator. If $T_1 \leq T_2$ then prove that $T_1 + T \leq T_2 + T$, where T, T_1, T_2 are bounded self adjoint linear operators on the same Hilbert space.
9. If T_1 and T_2 are two positive operators on a complex Hilbert space H, then prove that
   (a) $T_1 + T_2 \geq 0$ and
   (b) $T_1 \leq T_2 \Rightarrow T_2 - T_1 \geq 0$.
10. Prove that all the eigen values (if they exist) of a bounded self adjoint linear operator $T : H \to H$ on a complex Hilbert space H are real.

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. (a) Define contraction on a metric space X.
    (b) Prove that a contraction on a metric space X is continuous.
12. Let $T : X \to Y$ be a bijective bounded linear operator from a Banach space X to a Banach space Y. Show that there exists two positive numbers a and b such that

$$a||x|| \leq ||Tx|| \leq b||x|| \text{ for all } x \in X.$$

13. State and prove the representation theorem of the resolvent operator $R_\lambda(T)$ where $T \in B(X, X)$, X is a complex Banach space.

14. Let A be a complex Banach algebra with identify `e'
(a) When we say an $x \in A$ is invertible ?
(b) Prove that the set `G' of all invertible elements of A is a multiplicative group.
(c) Prove that 'G' is an open subset of A.

15. Let X and Y be two normed spaces.
(a) Prove that $c(X, Y)$ is a subspace of $B(X, Y)$.
(b) Prove that $c(X, Y)$ is a closed subspace of $B(X, Y)$, if Y is a Banach space.
(c) Prove that $c(X, Y)$ is a Banach space, if Y is a Banach space.

16. Prove that the positive square root of a bounded positive self adjoint linear operator on a complex Hilbert space H is unique.

(5 x 4 = 20)

PART C

Answer any 4 (10 marks each)

17.1. (a) Prove that uniform operator convergence of a sequence (T_n) of bounded linear operators implies strong operator convergence of (T_n) with the same limit, but the converse is not true.
(b) Prove that strong operator convergence of a sequence (T_n) in $B(X, Y)$ implies weak operator convergence of (T_n) with the same limit but the converse is not true.

OR

2. (a) Suppose $T : \mathcal{D}(T) \to Y$ is linear where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces.
(i) Define graph $\mathcal{G}(T)$ of T
(ii) Prove that $\mathcal{G}(T)$ is a subspace of $X \times Y$
(iii) Prove that $X \times Y$ is a normed space under the norm defined by $||(x, y)|| = ||x|| + ||y||$
(iv) If X and Y are Banach spaces, prove that $X \times Y$ is also a Banach space and $\mathcal{G}(T)$ is a closed subspace of $X \times Y$
(b) Give an example of a closed linear operator which is not bounded and a bounded linear operator which is not closed.

18.1. Let A be a complex Banach algebra with identity `e'. Then for any $x \in A$ prove that $\sigma(x)$ is compact. Also define the spectral radius $r_\sigma(x)$ of x and prove that $r_\sigma(x) \leq ||x||$
(b) Let A be a complex Banach algebra with identity `e'. Prove that for any $x \in A$, $\sigma(x) \neq \phi$.

OR

2. (a) Let A be a complex Banach algebra with identity `e'. If $x \in A$ is such that $||x|| < 1$, then prove that $(e - x)$ is invertible and

$$(e - x)^{-1} = \sum_{j=0}^{\infty} x^j \quad (x^0 = e).$$

Hence show that $||(e - x)^{-1} - e - x|| \leq \frac{||x||^2}{1 - ||x||}$

(b) If $||x - e|| < 1$, show that x is invertible and $x^{-1} = e + \sum_{j=1}^{\infty}(e-x)^j$

(c) If $x \in A$ is invertible and $y \in A$ is such that $\|yx^{-1}\| < 1,$ show that $x - y$ is invertible and $(x-y)^{-1} = \sum_{j=0}^{\infty} x^{-1}(yx^{-1})^j$.

19.1. (a) Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then:

(i) If T is bounded and $\dim T(X) < \infty$, then prove that T is compact.

(ii) If $\dim X < \infty$, then prove that T is compact.

(b) Let $T : l^2 \to l^2$ be defined by $Tx = y$, when $x = (\xi_j) \in l^2$ and $y = (n_j)$, $n_j = \frac{\xi_j}{j}$ for $j = 1, 2, 3, \ldots$. Then prove that T is compact linear.

OR

2. Let $T : X \to X$ be a compact linear operator and let $\lambda \neq 0$ (X is a normed space). Then prove that there exists a smallest integer $n = r$ such that,

$$N(T_\lambda^r) = N(T_\lambda^{r+1}) = \ldots \text{ and } T_\lambda^r(X) = T_\lambda^{r+1}(X) = \ldots$$

and if $r > 0$, the following inclusions are proper,

$$N(T_\lambda^0) \subset N(T_\lambda) \subset N(T_\lambda^2) \subset \ldots \subset N(T_\lambda^r)$$

and

$$T_\lambda^0(X) \supset T_\lambda(X) \supset \ldots \supset T_\lambda^r(X).$$

20.1. Suppose P_1 and P_2 are two projections on a Hilbert space H

(a) Prove that $P = P_2 - P_1$ is a projection on H if and only if $Y_1 \subset Y_2$ where $Y_j = P_j(H)$, $j = 1, 2$

(b) If $P = P_2 - P_1$ is a projection, then prove that P projects H onto Y, where Y is the orthogonal complement of Y_1 in Y_2.

OR

2. Let P_1 and P_2 be two projections on a Hilbert space H

(a) Prove that $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal

(b) If $P = P_1 + P_2$ is a projection, prove that P projects H onto $Y = Y_1 \oplus Y_2$.

(10 x 4 = 40)