Reg. No .... Name **23P2055-S**

MSc DEGREE END SEMESTER EXAMINATION - MARCH 2023

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT10 ; REAL ANALYSIS**

(For Supplementary 2020/2019/2018/2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer all (1.5 marks each)

1. Prove that $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$.
2. Let f and g be two functions of bounded variation on [a,b]. Prove that fg is of bounded variation on [a,b].
3. Discuss the uniform convergence of $\{f_n\}$, where $f_n(x) = x^n$; $x \in$[0,1].
4. If $f_1, f_2 \in \mathscr{R}(\alpha)$ on [a,b], then prove that $f_1 + f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$.
5. Define Uniform closure. Prove that a sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \longrightarrow f$ uniformly on X.
6. If f is continuous on [a,b], then prove that $f \in \mathscr{R}(\alpha)$.
7. Prove that $\lim_{x \to 0}(1+x)^{1/x} = e$.
8. Distinguish between pointwise and uniform convergence of a sequence of functions.
9. Does there exist a function which is neither of bounded variation nor Riemann Integrable. Justify.
10. Let $\{a_{ij}\}, i = 1, 2, 3, \ldots; j = 1, 2, 3, \ldots$ be a double sequence. If $\sum_{j=1}^{\infty} |a_{ij}| = b_i, i = 1, 2, 3, \ldots$ and $\sum b_i$ converges, prove that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. Suppose $\{f_n\}$ is a sequence of functions defined on E and there exist sequences of real numbers $\{a_n\}$ and $\{x_n\} \in E$ such that $|f_n(x_n) - f(x_n)| > a_n, \ \forall n$. Then prove that f_n does not converge uniformly to f, if $a_n \nrightarrow 0$.
12. State and prove the fundamental theorem of calculus.
13. Prove that $f \in \mathscr{R}(\alpha)$ on [a,b] iff for every $\epsilon > 0$, there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.
14. Find the total variation of the function $f(x) = \sin 2x$ over the interval $[0, 2\pi]$.
15. If $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, prove that $E(x) = e^x \ \forall x \in \mathbb{R}$
16. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that $\lim_{n \to \infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$.

(5 x 4 = 20)

PART C

Answer any 4 (10 marks each)

17.1 a. Let f be a function of bounded variation on [a,b]. If $c \in [a, b]$, prove that f is a function of bounded variation on [a,c] and [c,b].

b. Find the total variation of the function $f(x) = |x^2 - 4|$ over [-4,4].

OR

2 a) Define the total variation of a function of bounded variation. Prove that the total variation is zero iff f is constant.

b) State and prove the additive property of total variation of a function of bounded variation.

18.1 Let $f(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$, for all $x \in$ [0,1], where C is the Cantor set. Prove that $f \in \mathscr{R}$ on [0,1].

OR

2\. 1. Prove that if $f \in \mathscr{R}(\alpha)$, $\int\limits_a^b f d\alpha = \int\limits_a^c f d\alpha + \int\limits_c^b f d\alpha$ for $a < c < b$.

2\. Evaluate $\int\limits_1^{10} f d\alpha$ where $f(x) = [logx]$ and α is the identity function.

19.1 If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, Prove that $\{f_n + g_n\}$ converges uniformly on E. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

OR

2\. Prove or disprove: uniform limit of a sequence of differentiable function is differentiable. State a sufficient condition for the uniform limit of a sequence of functions being differentiable

20.1 Suppose $\sum\limits_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and define $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n \quad (|x| < R)$, then prove that $\sum\limits_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R + \epsilon, R - \epsilon]$, no matter which $\epsilon > 0$ is chosen and the function f is continuous and differentiable in $(-R, R)$ with

$f'(x) = \sum\limits_{n=0}^{\infty} n c_n x^{n-1}$.(

OR

2\. Introduce trigonometric functions using exponential series and hence derive any two properties of them. (10 x 4 = 40)