Reg. No .... Name **23P2055**

M. Sc. DEGREE END SEMESTER EXAMINATION : MARCH 2023

SEMESTER 2 : MATHEMATICS

COURSE : **21P2MATT10: MEASURE THEORY AND INTEGRATION**

(For Regular - 2022 Admission and Supplementary - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. If ϕ is a non -negative simple function and $A \supset B$, then prove that $\int_A \phi \geq \int_B \phi$. (A, CO 3)
2. If $m^*A = 0$, then prove that $m^*(A \cup B) = m^*B$ for any set B. (A, CO 2)
3. Define Borel sets. Is an open set a Borel set? Justify. (R, CO 1)
4. Define the characteristic function of a set A. If A and B are two sets, prove that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$. (A, CO 2)
5. Define a measure space. Give an example. (U, CO 4)
6. If $V \subset X \times Y$, then prove that $(\chi_V)_x = \chi_{V_x}$ and $(\chi_V)^y = \chi_{V^y}$. (A, CO 5)
7. If f is a non-negative measurable function and 'a' is a positive constant such that $f \geq a$ on a measurable set E, prove that $\int_E f \geq amE$. (A, CO 3)
8. If μ_1 and μ_2 are two measure on a measurable space (X, B) and a and b are two positive constants, then prove that $a\mu_1 + b\mu_2$ is also a measure. ()
9. If f is integrable, then prove that f is finite valued a.e. (A, CO 3)
10. Let `c' be a constant and f be a measurable function defined on X, where $(X, \mathcal{B})$ is a measurable space. Then prove that cf and $f + c$ are measurable. (U, CO 3)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Assume that $\langle E_i \rangle$ is a sequence of disjoint measurable sets and $E = \cup E_i$. Then for any set A, prove that ()

$$\mu^*(A \cap E) = \sum \mu^*(A \cap E_i).$$

12. Prove that a measurable function f is integrable over a measurable set E if and only if both f^+ and f^- are integrable over E. (A, CO 3)
13. (a) Show that every non-empty open set has a positive measure.
(b) If $\{A_n\}$ is a countable collection of sets of real numbers, then prove that $m^*(UA_n) \leq \sum m^* A_n$. (A, CO 2)
14. By integrating $e^{-y} \sin 2xy$ w.r.t x and y, show that $\int_0^\infty e^{-y} \frac{sin^2 y}{y} dy = \frac{log 5}{4}$. (A, CO 5)

15. Let ϕ and ψ be simple functions which vanish outside a set of finite measure,then prove that (A, CO 3)
$\int(a\phi + b\psi) = a\int\phi + b\int\psi$ where a and b are constants.
Hence, prove that
$\int(\phi - \psi) = \int\phi - \int\psi.$

16. Prove that m^* is translation invariant. (A)

17. Show that if f is $\mathcal{S}$ measurable and g is $\mathcal{J}$ measurable then fg is $\mathcal{S}$x$\mathcal{J}$ measurable. (A, CO 1, CO 2, CO 5)

18. Let $\mathfrak{A}$ be an algebra of subsets of a space X. If $A \in \mathfrak{A}$ and if $\langle A_i \rangle$ is any sequence of sets in $\mathfrak{A}$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$, then prove that ()

$$\mu A \leq \sum_{i=1}^{\infty} \mu A_i.$$

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. If $E \in \mathcal{S} \times \mathcal{J}$, then prove that for each $x \in X$ and $y \in Y$, $E_x \in \mathcal{J}$ and $E^y \in S$. (A, CO 4)

20. Prove that the outermeasure of an interval is its length. (A, CO 1)

21. State and prove any two convergence theorems. (A)

22. Let E be a measurable set such that $0 < \nu E < \infty$. Then prove that there is a positive set $A \subset E$ with $\nu A > 0$. (A)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	define measurable set, measurable function , Lebesgue integral and to relate Lebesgue integral with Riemann integral.	A	3, 17, 20	8
CO 2	explain the relevance of Lebesgue integration	A	2, 4, 13, 17	6
CO 3	solve problems related to Lebegue integral , Lebesgue and abstract measure, Lebesgue and abstract outer measure , signed measure ,Integral with respect to a measure , Integral with respect to product measure .	A	1, 7, 9, 10, 12, 15	8
CO 4	analyse the algebraic properties of Lebesgue integrable functions and Lebesgue measurable functions.	An	5, 19	6
CO 5	develop an algebraic as well as a geometrical structure for the collection of all integrable functions.	Cr	6, 14, 17	5

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;