Reg. No .... Name **22P1019**

M. Sc. DEGREE END SEMESTER EXAMINATION : OCTOBER 2022

SEMESTER 1 : MATHEMATICS

COURSE : **21P1MATT02: ALGEBRA**

(For Regular - 2022 Admission and Supplementary - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Give an example of an infinite torsion group. (An, CO 1)
2. Define torsion subgroup of an abelian group. Find the torsion subgroup of the multiplicative group R^* of nonzero real numbers. (U, CO 1)
3. Find the sum and product of $f(x) = 2x^2 + 3x + 4$ and $g(x) = 3x^2 + 2x + 3$ in $\mathbb{Z}_6[x]$. (U, CO 2)
4. For the evaluation homomorphism $\phi_4 : \mathbb{Z}_7[x] \to \mathbb{Z}_7$, evaluate $\phi_5[(x^3 + 2)(4x^2 + 3)(x^7 + 3x^2 + 1)]$. (E)
5. Find all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[x]/ < x^3 + x^2 + c >$ is a field? (A, CO 2)
6. Define a constructible number. (R, CO 3)
7. Find all irreducible polynomials of degree 2 in $\mathbb{Z}_2[x]$. (A, CO 3)
8. Show that $R[x]/\langle x^2 + 1\rangle \cong \mathbb{C}$. (A, CO 3)
9. True or False: $\mathbb{Q}(i)$ is a splitting field over $\mathbb{Q}$. Justify. (A, CO 4)
10. State the Conjugation Isomorphisms Theorem for Field Theory. (R, CO 4)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. Let G,H and K be finitely generated abelian groups. Show that if $G \times K$ is isomorphic to $H \times K$, then $G \cong H$. (A, CO 1)
12. Show that for a prime number p, every group of order p^2 is abelian. (An, CO 1)
13. Show that $x^4 - 22x + 1$ is irreducible over $\mathbb{Q}$. (E, CO 2)
14. If F is a field, prove that every non-constant polynomial $f(x) \in F[x]$ can be factored in $F[x]$ into a product of irreducible polynomials, the irreducible polynomials being unique except for order and for unit factors in F. Find the unique factorization of $x^4 + 3x^3 + 2x + 4$ in $\mathbb{Z}_5[x]$. (A, CO 2)
15. Show that the elements of $GF(p^n)$ are precisely the zeroes in $\overline{\mathbb{Z}_p}$ of the polynomial $x^{p^n} - x$ in $\mathbb{Z}_p[x]$. (An, CO 3)
16. Define algebraic and transcendental numbers. Show that the set of all algebraic numbers forms a field. (U, CO 3)
17. Show that if E is a finite extension of F, then $\{E : F\}$ divides $[E : F]$. (A, CO 4)

18. Give an example of two finite normal extensions K_1 and K_2 of the same field F such that K_1 and K_2 are not isomorphic but $G(K_1/F) \cong G(K_2/F)$. (E, CO 4)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. (a). Let X be a G-set.For $x_1, x_2 \in G$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Show that $\sim$ is an equivalence relation on X. What is the equivalence class of $x \in X$ under this equivalence relation known as?
(b). Let X be a G-set. Show that $G_x = \{g \in G \,|\, gx = x\}$ is a subgroup of G for each $x \in X$. What is this subgroup known as?
(c). Let G be a group of order p^n and let X be a finite G-set. Let $X_G = \{x \in X|\, gx = x \text{ for all } g \in G\}$. Show that $|X| \cong |X_G|(\mod p)$. (E, CO 1)

20. (a). State and prove Eisenstein's criterion for irreducibility over $\mathbb{Q}$.
(b). Show that the p^{th} cyclotomic polynomial is irreducible over $\mathbb{Q}$ for any prime p. (E, CO 2)

21. (a). Define a constructible number.Show that the set of all constructible numbers forms of a subfield F of the field of real numbers.
(b). Show that if γ is constructible and $\gamma \notin \mathbb{Q}$, show that $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2^r$ for some integer $r \geq 0$. (An, CO 3)

22. Find the splitting field K of $x^4 + 1$ over $\mathbb{Q}$. Compute $G(K/\mathbb{Q})$, find its subgroups and the corresponding fixed fields and draw the subgroup and subfield lattice diagrams. (E, CO 4)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Develod ideas of finitely generated abelian groups, Sylow theorems and applications.	E	1, 2, 11, 12, 19	11
CO 2	Explain the concept of rings of polynomials, factorisation of polynomials and ideal structure.	E	3, 5, 13, 14, 20	11
CO 3	Illustrate the idea of extension fields, algebraic extensions and geometric constructions.	E	6, 7, 8, 15, 16, 21	12
CO 4	Devrlop ideas of automorphisms of fields, isomorphism extension theorem and Galois theory.	E	9, 10, 17, 18, 22	11

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;