Reg. No .... Name **22P1004**

M. Sc. DEGREE END SEMESTER EXAMINATION : OCTOBER 2022

SEMESTER 1 : MATHEMATICS

COURSE : **21P1MATT01 : LINEAR ALGEBRA**

(For Regular - 2022 Admission and Supplementary - 2021 Admission)

Duration : Three Hours Max. Weights: 30

PART A

Answer any 8 questions **Weight: 1**

1. Find the rank of the linear transformation $T : R^3 \to R^2$ defined by $T(x, y, z) = (x, y)$. (An, CO 2)
2. Define annihilator of a subset S of a vector space V. What is the annihilator of $S = \{0\}$? (U, CO 2)
3. Show that there is at least one non-zero polynomial that annihilates a given linear operator on a finite-dimensional vector space. (A, CO 4)
4. Is the set of vectors $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $a_1 a_2 = 0$, a subspace of $\mathbb{R}^n$? (A, CO 1)
5. Define the terms characteristic value, characteristic vector and characteristic space with respect to a linear operator T on a vector space V. (U, CO 4)
6. Define commutative and non-commutative rings. Give examples for each. (U, CO 3)
7. Define a determinant function. (U, CO 3)
8. Define minimal polynomial for a linear operator T on a finite dimensional vector space V. State three properties which characterize the minimal polynomial. (U, CO 4)
9. Let V be the (real) vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$. Is the set of all functions f such that $f(-1) = 0$, a subspace of V? (U, CO 1)
10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_2, x_1)$. Is T linear? Explain (U, CO 2)

(1 x 8 = 8)

PART B

Answer any 6 questions **Weights: 2**

11. If W_1 and W_2 are subspaces of a finite-dimensional vector space, show that $W_1 = W_2$ if and only if $W_1^\circ = W_2^\circ$. (An, CO 2)
12. Let T be a linear operator on a finite dimensional vector space V. Show that the minimal polynomial divides the characteristic polynomial for T. (A, CO 4)
13. If A^t denotes the transpose of an $n \times n$ matrix A with complex entries, prove that det A^t = det A. (A, CO 3)

14. Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Show that T is diagonalizable if and only if the minimal polynomial for T is a product of linear polynomials over F. (A, CO 4)

15. Show that the vectors $\alpha_1 = (1,1,0,0), \alpha_2 = (0,0,1,1), \alpha_3 = (1,0,0,4)$ and $\alpha_4 = (0,0,0,2)$ form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. (A, CO 1)

16. Are the vectors $(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0)$ and $(2,1,1,6)$ linearly independent in R^4? Justify your answer. (U, CO 1)

17. Find the subspace of $\mathbb{R}^4$ annihilated by the three linear functionals
$f(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 2x_3 + x_4$,
$f(x_1, x_2, x_3, x_4) = 2x_2 + x_4$ and
$f(x_1, x_2, x_3, x_4) = -2x_1 - 4x_3 + 3x_4$. (A, CO 2)

18. Define hyperspace in a vector space. Give an example. (U, CO 3)

(2 x 6 = 12)

PART C

Answer any 2 questions **Weights: 5**

19. (a) Let T be a linear operator on a finite dimensional space V. Let $c_1, c_2, \cdots, c_k$ be the distinct characteristic values and $W_1, W_2, \cdots, W_k$ be the corresponding characteristic spaces. Prove that $\dim(W_1 + W_2 + \cdots + W_k) = \dim W_1 + \dim W_2 + \cdots + \dim W_k$.
(b) If W_1 and W_2 are subspaces of V then prove that they are independent if and only if $W_1 \cap W_2 = 0$. (An, CO 4)

20. a) Define rank and nullity of a linear transformation.
(b) Let V be finite dimensional and $T : V \to W$ be a linear transformation. Prove that $\text{Rank}\, T + \text{Nullity}\, T = \dim V$
(c) Determine a linear transformation from R^4 into R^3 which has its range the subspace spanned by (1,0,0) and (1,1,0). What is the Nullity of such a linear transformation? (A, CO 2)

21. (a) Let D be an n-linear function on the space of $n \times n$ matrices over a field K. Suppose D has the property that $D(A) = 0$ whenever two adjacent rows of A are equal. Show that D is alternating.
(b) Let $n > 1$ and let D be an alternating $(n-1)$ linear function on an $(n-1) \times (n-1)$ matrix over K. Show that for each $j, j = 1, \ldots, n$, the function E_j defined by $E_j(A) = \sum_{i=1}^{n} (-1)^{(i+j)} A_{ij} D_{ij}(A)$ is an alternating n-linear function on the space of $n \times n$ matrices A. If D is the determinant function, so is E_j. (An, CO 3)

22. Let V be an n-dimensional vector space over the field F and let $\mathscr{B}$ and $\mathscr{B}'$ be two ordered bases of V.Show that there is a unique necessarily invertible $n \times n$ matrix P with entries in F such that $[\alpha]_{\mathscr{B}} = P[\alpha]_{\mathscr{B}'}$ and $[\alpha]_{\mathscr{B}'} = P^{-1}[\alpha]_{\mathscr{B}}$. (An, CO 1)

(5 x 2 = 10)

OBE: Questions to Course Outcome Mapping

CO	Course Outcome Description	CL	Questions	Total Wt.
CO 1	Summarize the concepts of vector spaces, subspaces, basis and dimension, coordinates and properties of row equivalence	U	4, 9, 15, 16, 22	11
CO 2	Explain the linear transformations and their algebra and representation of transformations by matrices.	An	1, 2, 10, 11, 17, 20	12
CO 3	Demonstrate the ideas of commutative Rings, determinant functions, permutation and uniqueness of determinants, additional properties of determinants	An	6, 7, 13, 18, 21	11
CO 4	Illustrate the ideas of characteristic values, annihilating polynomials, invariant subspace, direct sum decomposition	An	3, 5, 8, 12, 14, 19	12

Cognitive Level (CL): Cr - CREATE; E - EVALUATE; An - ANALYZE; A - APPLY; U - UNDERSTAND; R - REMEMBER;