Reg. No .... Name **23U604**

B. Sc. DEGREE END SEMESTER EXAMINATION : MARCH 2023

SEMESTER 6 : MATHEMATICS

COURSE : **19U6CRMAT09 : REAL ANALYSIS - 2**

(For Regular - 2020 Admission and Supplementary - 2019 Admission)

Time : Three Hours Max. Marks: 75

PART A

Answer any 10 (2 marks each)

1. Define discontinuity of the second kind of a function f.
2. Define discontinuity of the first kind from the left of a function f.
3. Define the Dirichlet's function.
4. Is every Riemann integrable function continuous? Justify your answer.
5. If n is a non-negative integer, prove that $\Gamma(n+1) = n!$.
6. Discuss the kind of discontinuity, if any, of the function
$$f(x) = \begin{cases} \dfrac{x-|x|}{x} & when\ x \neq 0 \\ 2 & when\ x = 0 \end{cases}$$
7. When is a partition P^* of $[a, b]$ said to be finer than another partition P of $[a, b]$?
8. Show that the series $\sum r^n \sin(a^n\theta),\ 0 < r < 1$, converges uniformly for all real values of θ.
9. Show that the series $\sum \dfrac{\sin(x^2+n^2x)}{n(n+1)}$, converges uniformly for all real x.
10. Compute $\Gamma(-\frac{9}{2})$.
11. State and prove the symmetrical property of the Beta function.
12. Define uniform convergence of a sequence of functions $\{f_n\}$.

(2 x 10 = 20)

PART B

Answer any 5 (5 marks each)

13. Show that a bounded function f, having a finite number of points of discontinuity on $[a, b]$ is integrable on $[a, b]$.
14. Show that if f is bounded and integrable on $[a, b]$ and k is a number such that $|f(x)| \leq k$ for all $x \in [a, b]$, then $|\int_a^b f\,dx| \leq k(b-a)$.
15. Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{x}{1+nx^2}$ is uniformly convergent on any closed interval.
16. Test for uniform convergence of the series
$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots,\ -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}$$
17. State and prove the intermediate value theorem.
18. Show that $B(m,n) = \displaystyle\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}},\ m > 0,\ n > 0.$

19. Discuss the convergence of $\int_0^2 \frac{dx}{2x - x^2}$.

20. Show that the function $f(x) = 1/x$ is not uniformly continuous on $(0, 1]$.

(5 x 5 = 25)

PART C

Answer any 3 (10 marks each)

21. Show that the sequence $\{f_n\}$, where $f_n(x) = x^n$ is uniformly convergent on $[0, k]$, where $k < 1$ and is pointwise convergent on $[0, 1]$.

22. Prove that a necessary and sufficient condition for the integrability of a bounded function f is that to every $\epsilon > 0$, there corresponds $\delta > 0$ such that for every partition P of $[a, b]$ with norm $\mu(P) < \delta, U(P, f) - L(P, f) < \epsilon$.

23. (i) If $n > 1$ is a positive integer, show that $B(m, n) = \frac{(n-1)!}{m(m+1)(m+2)\ldots(m+n-2)(m+n-1)}$.

    (ii) If $m > 1$ is a positive integer, show that $B(m, n) = \frac{(m-1)!}{n(n+1)(n+2)\ldots(n+m-2)(n+m-1)}$.

    (iii) If $m > 1$ and $n > 1$ are positive integers, prove that $B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$.

24. Show that if a function f is continuous on a closed interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one point $\alpha \in (a, b)$ such that $f(\alpha) = 0$.

(10 x 3 = 30)