Reg. No……………………………………………. Name……………………………………………

B.Sc. DEGREE END SEMESTER EXAMINATION - APRIL 2021

SEMESTER –6: MATHEMATICS (CORE COURSE)

COURSE: 15U6CRMAT12: LINEAR ALGEBRA AND METRIC SPACES

(Common for Regular 2018 admission & Improvement 2017/Supplementary 2017/2016 /2015 admissions)

Time: Three Hours Max Marks: 75

PART A

Answer all questions. Each question carries 1 mark

1. Define linear independence and linear dependence of a vector space.
2. Show that subset of a vector space consisting of the single vector **zero** is linearly dependent?
3. Define basis of a vector space.
4. If $L: R^n \to R^m$ is defined as $L(u) = \mathrm{Au}$ for an $\mathrm{m} \times \mathrm{n}$ matrix A then show that L is linear.
5. Define kernel of a linear transformation with an example.
6. Define nullity and rank of a linear transformation.
7. Define metric on a non-empty set X and write an example for that.
8. Define open set and closed set in a metric space.
9. "A cauchy sequence is not necessarily convergent". Clarify the statement with an example.
10. Define Uniformly continuous function. Give an example of a function which is continuous but not uniformly continuous. (1 x 10 = 10)

PART B

Answer any Eight questions. Each question carries 2 marks

11. Determine whether $u = [1 \quad 2 \quad 3]$ is a linear combination of $v_1 = [1 \quad 1 \quad 1]$, $v_2 = [2 \quad 4 \quad 0]$ and $v_3 = [0 \quad 0 \quad 1]$?
12. Determine whether the set $\{[1 \quad 1 \quad 0], [0 \quad 1 \quad 1], [1 \quad 0 \quad 1]\}$ form a basis for R^3?
13. Determine the row rank of $\mathrm{A}=\begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix}$.
14. Determine whether the transformation T is linear if $T: R^2 \to R^2$ is defined by $T[\mathrm{a\ b}]=[\mathrm{a\ -b}]$ all real numbers a and b.

15. A Linear transformation $T: R^2 \to R^2$ has the property that $T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}$ and $T\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}7\\8\end{bmatrix}$.

    Determine Tv for any vector $v \in R^2$.
16. Show that image of a linear transformation $T: V \to W$ is a subspace of W .
17. Prove that Cantor set is closed.
18. Let **X** be a complete metric space and **Y** be a subspace of **X**. Prove that **Y** is complete iff it is closed.
19. Show that a Cauchy sequence is convergent iff it has a convergent sub sequence.
20. Let **X** be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in **X** such that $x_n \to x$ and $y_n \to y$ then show that $d(x_n, y_n) \to d(x, y)$. (2 x 8 = 16)

PART C

Answer any Five questions. Each question carries 5 marks

21. Prove that If $S=\{v_1, v_{2,} ...., v_n\}$ is a basis for a vector space **V** then any set containing more than n vectors is linearly dependent.
22. Find a basis for the span of the vectors in $\{t^3+3t^2, 2t^3+2t-2, t^3-6t^2+3t-3, 3t^2-t+1\}$.
23. Find the matrix representation for the linear transformation $T: M_{2\times2} \to M_{2\times2}$ defined by

    $T\begin{bmatrix}\text{a} & \text{b}\\ \text{c} & \text{d}\end{bmatrix}=\begin{bmatrix}\text{a+2b+3c} & \text{2b-3c+4d}\\ \text{3a-4b-5d} & 0\end{bmatrix}$ with respect to the standard basis

    $B=\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$
24. Identify the kernel and the image of the linear transformation $T: P^2 \to M_{2\times2}$ defined by

    $T(\text{a}t^2+bt+c)=\begin{bmatrix}\text{a} & \text{2b}\\ 0 & \text{a}\end{bmatrix}$ for all real numbers a, b and c.
25. Let **X** be a metric space with metric **d**. Show that d_1 defined by $d_1(x, y)=\dfrac{d(x, y)}{1+d(x, y)}$ is a metric on **X**.
26. Let **X** be a metric space. Then prove that a subset **G** of **X** is open iff it is a union of open spheres.

27. Let ***X*** be a complete metric space and let ***Y*** be a subspace of ***X***. Prove that ***Y*** is complete if and only if ***Y*** is closed. (5 x 5 =25)

PART D

Answer any Two questions. Each question carries 12 marks

28. (a) Determine whether $S = \{[x \quad y \quad z] \in R^3 | y = 0\}$ is a vector space under regular addition and scalar multiplication.
    (b) Prove that additive inverse of any vector v in a vector space V is unique?

29. Let **T** be a linear transformation from an n-dimentional vector space **V** into **W** and let $\{v_1, v_2, ..., v_k\}$ be a basis for the kernel of T. If this basis is extended to a basis $\{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$ for V, then prove that $\{T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}$ is a basis for the image of T.
30. Let X be metric space then prove the following: (a) the empty set Φ and the full space X are closed Sets (b) a subset F of X is closed iff its complement F' is open (c) each closed sphere is a closed set.
31. (a) State and prove Cantor's Intersection Theorem
    (b) Let X be a metric space, Y be a complete metric space and A a dense subspace of X. If f is uniformly continuous mapping of A into Y then prove that f can be extended uniquely to uniformly continuous mapping g of X into Y. (12 x 2 = 24)

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