Reg. No .... Name **17P3653**

MSc DEGREE END SEMESTER EXAMINATION- OCTOBER-NOVEMBER 2017

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT15 ; NUMBER THEORY**

(For Regular - 2016 admission)

Time : Three Hours Max. Marks: 75

Section A

Answer any 10 (1.5 marks each)

1. Show that the divisor functions are multiplicative.
2. Let f be an arithmetical function with $f(1)=1$. Prove that f is multiplicative if and only if $f\left(p_1^{a_1}\ldots p_k^{a_k}\right)=f\left(p_1^{a_1}\right)\ldots f\left(p_k^{a_k}\right)$.
3. Assume $(a,m)=d$. Prove that the linear congruence $ax\equiv b(\mod m)$ has solutions, if and only if, $d|b$.
4. State and prove Wilson's theorem.
5. Let D be a domain. Prove that x is a unit if and only if $x|1$.
6. Let D be a domain. Prove that x is irreducible if and only if every divisor of x is an associate of x or a unit.
7. Prove that factorization into irreducible is possible in $\mathfrak{O}$.
8. Let $\mathfrak{a},\mathfrak{b}$ be two ideals of $\mathfrak{O}$. Prove that $\mathfrak{a}|\mathfrak{b}$ iff $\mathfrak{a}\supseteq\mathfrak{b}$
9. If $\mathfrak{a}\neq 0$ is an ideal of $\mathfrak{O}$ with $\mathrm{N}(\mathfrak{a})$ is prime , prove that $\mathfrak{a}|\mathrm{N}(\mathfrak{a})$
10. Prove that $\mathbb{R}[x,y]/\langle x\rangle$ is isomorphic(as rings) to $\mathbb{R}[y]$.

10 x 1.5 (15)

Section B

Answer any 4 (5 marks each)

11. Derive the formula for divisor sum of $\lambda(n)$.
12. Prove that the set of lattice points visible from the origin has density $6/\pi^2$
13. Find all integers n such that $\varphi(n)=12$
14. Prove that for $x\geq 2$, $\pi(x)=\dfrac{\vartheta(x)}{\log x}+\displaystyle\int_2^x\frac{\vartheta(t)}{t\log^2 t}dt$.
15. Prove that every Euclidean domain is a unique factorization domain.
16.

Let $\mathfrak{a}$ be an ideal of $\mathfrak{O}$. Prove that $\mathrm{N}(\mathfrak{a}) = \left|\frac{\Delta[\alpha_1, \ldots, \alpha_n]}{\Delta}\right|$, where Δ is the discriminant.

4 x 5 (20)

Section C
Answer any 4 (10 marks each)

17.1. Let $\mathscr{F} = \{f : \mathbb{N} \to \mathbb{C} |\ f(1) \neq 0\}$. Prove that $\mathscr{F}$ is a group under Dirichlet multiplication.

OR

2. Prove that the average order of $\varphi(n)$ is $3n/\pi^2$.

18.1. 1. Prove the converse of Wilson's theorem.
2. Find all positive integers n for which $(n-1)! + 1$ is a power of n.

OR

2. Prove that for every integer $n \geq 2$, $\frac{1}{6} \frac{n}{\log n} < \pi(n) < 6\frac{n}{\log n}$

19.1. Define Euclidean quadratic Field. Prove that the ring of integers $\mathfrak{O}$ of $\mathbb{Q}(\sqrt{d})$ is Euclidean for $d = -2, -11$.

OR

2. If R is Noetherian, prove that $R[x]$ is Noetherian.

20.1. Prove that factorization of elements of $\mathfrak{O}$ into irreducibles is unique if and only if every ideal of $\mathfrak{O}$ is principal.

OR

2. Find all the ideals in $\mathbb{Z}[\sqrt{-5}]$ which contain the element 6.

4 x 10 (40)