Reg. No .... Name **17P3618**

MSc DEGREE END SEMESTER EXAMINATION- OCTOBER-NOVEMBER 2017

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT12 ; ADVANCED FUNCTIONAL ANALYSIS**

(For Regular - 2016 admission)

Time : Three Hours Max. Marks: 75

Section A

Answer any 10 (1.5 marks each)

1. Prove that every weakly convergent sequence is bounded.
2. Define strong and weak* convergence of a sequence of bounded linear functionals on a normed space.
3. If X and Y are normed spaces, prove that $X \times Y$ is a normed space under the norm defined by $||(x, y)|| = ||x|| + ||y||$.
4. Define Banach algebra. Given an example
5. Let A be a Banach algebra with identity. Then prove that the identity element of A is unique.
6. Define compact linear operator.
7. Let $T_j : X_j \to X_{j+1}, j = 1, 2, 3$ be bounded linear operators. If T_2 is compact, show that $T = T_3T_2T_1 : X_1 \to X_4$ is compact.
8. Define a positive operator on a complex Hilbert space.
9. If T_1 and T_2 are two positive operators on a complex Hilbert space H, then prove that
(a) $T_1 + T_2 \geq 0$ and
(b) $T_1 \leq T_2 \Rightarrow T_2 - T_1 \geq 0$.
10. Find a linear operator $T : R^2 \to R^2$ which is both idempotent and self adjoint.

10 x 1.5 (15)

Section B

Answer any 4 (5 marks each)

11. Let $X = \{x \in R | x \geq 1\} \subset R$ and let the mapping $T : X \to X$ be defined by $Tx = \frac{x}{2} + x^{-1}$. Show that T is a contraction and find the smallest value of α in the condition of contraction.
12. Prove that every strongly convergent sequence is weakly convergent. Is the converse true?. Justify.
13. Let A be a complex Banach algebra with identify \`e'
(a) When we say an $x \in A$ is invertible ?

(b) Prove that the set `G' of all invertible elements of A is a multiplicative group.
(c) Prove that 'G' is an open subset of A.

14. Let A be a Banach algebra without identity. If we define $\tilde{A} = \{(x, \alpha) | x \in A, \alpha \text{ is a scalar}\}$, then prove that $\tilde{A}$ is a Banach algebra with identity under the following operations

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta),$$
$$\beta(x, \alpha) = (\beta x, \beta\alpha), (x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$
$$||(x, \alpha)|| = ||x|| + |\alpha|$$

15. Let B be a subset of a metric space X.
(a) If B is totally bounded, that prove that for any $\epsilon > 0$, B has a finite ϵ-net $M_\epsilon \subset B$.
(b) If B is totally bounded, then prove that B is separable.

16. Let P_1 and P_2 be projections on a Hilbert space H and let $P_1(H) = Y_1$ and $P_2(H) = Y_2$. Then prove that the following conditions are equivalent.
(a) $P_2P_1 = P_1P_2 = P_1$
(b) $Y_1 \subset Y_2$
(c) $N(P_1) \supset N(P_2)$
(d) $||P_1x|| \leq ||P_2x||$ for all $x \in H$
(e) $P_1 \leq P_2$.

4 x 5 (20)

Section C
Answer any 4 (10 marks each)

17.1. (a) State and prove closed graph theorem.
(b) Give an example of a closed linear operator which is not bounded linear.
OR

2. (a) Let X and Y be normed spaces and X compact. If $T : X \to Y$ is a bijective closed linear operator, show that T^{-1} is bounded.
(b) Show that the null space $N(T)$ of a closed linear operator $T : X \to Y$ is a closed subspace of X.
(c) Let X and Y be normed spaces. If $T_1 : X \to Y$ is closed linear and $T_2 : X \to Y$ is bounded linear, prove that $T_1 + T_2$ is a closed linear operator.

18.1. (a) If X is a non-zero complex Banach space and $T \in B(X, X)$, then prove that $\sigma(T) \neq \phi$.
(b) If $T \in B(X, X)$, where X is a non-zero complex Banach space, then prove that.

$$r_\sigma(T) = \lim_{n\to\infty} \sqrt[n]{||T^n||}$$

OR

2. (a) Let $T : X \to X$ be a bounded linear operator on a complex Banach space X.

Prove that for any $\lambda_0 \in \rho(T)$, $R_\lambda(T)$ has the representation

$R_\lambda(T) = \sum_{j=0}^{\infty}(\lambda - \lambda_0)^j R_{\lambda_0}(T)^{j+1}$ and the series is absolutely convergent

within the open disc given by $|\lambda - \lambda_0| < \frac{1}{||R_{\lambda_0}(T)||}$

(b) Let $T \in B(X, X)$ where X is a complex Banach space. If $\lambda, \mu \in \rho(T)$, prove the following

(i) $R_\mu - R_\lambda = (\mu - \lambda)R_\mu R_\lambda$

(ii) R_λ commutes with any $S \in B(X, X)$ satisfying $ST = TS$

(iii) $R_\lambda R_\mu = R_\mu R_\lambda$.

19.1. (a) Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent, then prove that the uniform operator limit is compact.

(b) Prove that the above result need not hold if we replace the uniform operator convergence by strong operator convergence.

OR

2. Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then prove that there exists a smallest integer r (depending on λ) such that from $n = r$ on the null spaces $N(T_\lambda^n)$ are all equal and if $r > 0$, the inclusion

$$N(T_\lambda^0) \subset N(T_\lambda) \subset N(T_\lambda^2) \subset \ldots \subset N(T_\lambda^r)$$

are all proper.

20.1. Let (P_n) be a monotone increasing sequence of projections P_n on a Hilbert space H.

(a) Show that (P_n) is strongly operator convergent $(P_n x \to P_x \forall x \in H)$ and the limit operator P is a projection on H

(b) Prove that $P(H) = \overline{\bigcup_{n=1}^{\infty} P_n(H)}$

(c) Prove that $N(P) = \bigcap_{n=1}^{\infty} N(P_n)$

OR

2. (a) Prove that the spectrum $\sigma(T)$ of a bounded self adjoint linear operator $T : H \to H$ on a complex Hilbert space H lies in the interval $[m, M]$, where

$$m = \inf_{||x||=1} \langle Tx, x\rangle \text{ and } M = \sup_{||x||=1} \langle Tx, x\rangle.$$

(b) Let $T : H \to H$ be a bounded self adjoint linear operator on a non-zero complex Hilbert space H. Then prove that $m, M \in \sigma(T)$.

4 x 10 (40)