Reg. No .... Name **17P131**

M Sc DEGREE END SEMESTER EXAMINATION- NOVEMBER 2017

SEMESTER 1 : MATHEMATICS

COURSE : **16P1MATT03 ; MEASURE THEORY AND INTEGRATION**

(Common for Regular - 2017 / Supplementary - 2016 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer all the questions (1.5 marks each)

1. Define a σ-algebra of subsets of a set X. Is the power set of X a σ-algebra of subsets of X?. Justify.
2. Define the extended real number system.
3. Give an example of a decreasing sequence $< E_n >$ of measurable sets such that
$$m\left(\cap_1^\infty E_n\right) \neq \lim mE_n.$$
4. Define the positive part and negative part of a function.
5. Prove that a measurable function f is integrable over a measurable set E if and only if both f^+ and f^- are integrable over E.
6. If f is a non-negative measurable function and 'a' is a positive constant such that $f \geq a$ on a measurable set E, prove that
$\int_E f \geq amE$.
7. Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $A, B \in \mathcal{B}$ and $A \subset B$. Then prove that $\mu A \leq \mu B$.
8. Define a positive set, a negative set and a null set with respect to a signed measure.
9. Let (X, B, μ) be a measure space and f be a non-negative measurable function defined on X. Prove that the set function ϕ defined as B by $\phi(E) = \int_E f d\mu$ is a measure.
10. Prove that the representation of a rectangle in the form $A \times B$ need not be unique.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. Let A be any set and E_1, E_2,. . .,E_n be a finite collection of disjoint measurable sets. Then prove that
$$m^*\left(A \cap (\bigcup_{i=1}^n E_i)\right) = \sum_{i=1}^n m^*(A \cap E_i).$$
Hence (and not otherwise), prove that
$$m^*(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n m^* E_i.$$

12. (a) Prove that χ_A is measurable if and only if A is measurable.
(b) Prove that the set of all points on which a sequence $\langle f_n \rangle$ of measurable functions converges is measurable.

13. Let f be a non-negative measurable function and $\langle E_i \rangle$ be a disjoint sequence of measurable sets. Let $E = \cup E_i$. Then prove that

$$\int_E f = \sum \int_{E_i} f.$$

14. a. If ϕ is a simple function taking the distinct values $a_1, a_2, \ldots, a_n$ on the disjoint measurable sets $A_1, A_2, \ldots, A_n$ respectively, then state the canonical representation of ϕ.
b. If E is any measurable set, prove that

$$\int_E \phi = \sum_1^n a_i \; m(A_i \cap E)$$

Using it prove that

$$\int_{A \cup B} \phi = \int_A \phi + \int_B \phi$$

if A and B are two disjoint measurable sets.

15. Let μ be a σ-finite measure on an algebra $\mathfrak{A}$ and let μ^* be the outer measure generated by μ. Prove that a set E is μ^*-measurable if and only if E is the proper difference $A - B$ of a set A in $\mathfrak{A}_{\sigma\delta}$ and a set B with $\mu^* B = 0$.
Each set B with $\mu^* B = 0$ is contained in a set C in $\mathfrak{A}_\sigma \delta$ with $\mu^* C = 0$.

16. If $\{A_i\}$ is a monotone sequence of subsets of $X \times Y$, then prove that $\lim A_i^y = (\lim A_i)^y$ and $\lim (A_i)_x = (\lim A_i)_x$, for each $x \in X$ and $y \in Y$.

(5 x 4 = 20)

Section C

Answer either 1 OR 2 of each question (10 marks each)

17.1. (a) Prove that the collection $\mathcal{M}$ of all measurable sets is a σ-algebra.
(b) Prove that (a, ∞) is measurable for all $a \in R$.

OR

2. (a) If f and g are two real valued measurable functions with the same domain, then
(i) Prove that $f + g$ is measurable.
(ii) Prove that cf is measurable, if c is a constant. Hence prove that $af + bg$ is measurable, if a and b are two constants. Deduce that $f - g$ is measurable.
(b) If f is a real valued measurable function defined on $(-\infty, \infty)$ and g is a continuous function, then prove that $g \circ f$ is measurable.

18.1. (a) State and prove Monotone Convergence theorem.
(b) State and prove Lebesgue Convergent theorem.

OR

2. (a) If f and g are non-negative measurable functions, prove the following

(i) $\int_E cf = c\int_E f, c > 0$
(ii) $\int_E (f+g) = \int_E f + \int_E g$
(iii) If $f \leq g$ a.e., then

$$\int_E f \leq \int_E g.$$

(b) Let $\langle f_n \rangle$ be a sequence of non-negative measurable functions that converge to f and suppose $f_n \leq f$ for all $n.$ Then prove that $\int f = \lim \int f_n.$

19.1. (a) Let f be an extended real valued function defined on X, where $(X, \mathcal{B})$ is a measurable space. Then prove that
the following statements are equivalent:
(i) $\{x \in X : f(x) < \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(ii) $\{x \in X : f(x) \leq \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(iii) $\{x \in X : f(x) > \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(iv) $\{x \in X : f(x) \geq \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(b) If μ is a complete measure and f is a measurable function, then prove that $f = g$ a.e. implies g is measurable.

OR

2. (a) Let (X, B, μ) be a measuer space and f be a measurable function defined on X such that $\int f d\mu$ is defined. Prove that the set function ν defined on B by $\nu E = \int_E f d\mu$ is a signed measure.
(b) Find a Hahn decomposition of X w.r.t. ν
(c) Find a Jordan decomposition of ν.

20.1. If $\mathcal{A}$ is an algebra, then prove that

$$S(\mathcal{A}) = \mathcal{M}_{\circ}(\mathcal{A})$$

OR

2. Let $[[X, \mathcal{S}, \mu]]$ and $[[Y, \mathcal{J}, v]]$ be σ-finite measure spaces. For $V \in \mathcal{S} \times \mathcal{J}$, write $\phi(x) = \nu(V_x)$ and $\psi(y) = \mu(V^y)$ for all $x \in X$ and $y \in Y$. Then prove that ϕ is $\mathcal{S}$-measurable and ψ is $\mathcal{J}$-measurable and $\int_X \phi d\mu = \int_Y \psi d\nu$.

(10 x 4 = 40)