Reg. No .... Name **21P4003**

M. Sc. DEGREE END SEMESTER EXAMINATION - APRIL 2021

SEMESTER 4 : MATHEMATICS

COURSE : **16P4MATT16EL : DIFFERENTIAL GEOMETRY**

(For Regular - 2019 Admission & Supplementary - 2018/2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer any 10 (1.5 marks each)

1. Describe the graphs and level sets(level curves) of $f(x_1, x_2) = x_1$.
2. Define Smooth Vector Field.
3. Sketch the vector field on $\mathbb{R}^2$: $\mathbb{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (x_2, -x_1)$.
4. Find the velocity, the acceleration, and the speed of parametrized curve $\alpha(t) = (t, t^2)$
5. Define covariant derivative of a parallel vector field.
6. Define Gauss map
7. Define parametrization of a segment of the plane curve C containing p.
8. Define the length of a parameterised curve.
9. Let f and g be two smooth functions on the open set $J \subset R^{n+1}$ show that $d(f+g) = df + dg$.
10. State inverse function theorem for n-surface.

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. Find the integral curve through $p = (x_1, x_2) = (1, 1)$ of the vector field $\mathbb{X}(p) = (p, x_2, -x_1)$.
12. State and prove the existence of Lagrange multiplier.
13. Let S be a 2-surface in $\mathbb{R}^3$ and let $\alpha : I \rightarrow S$ be a geodesic in S with $\dot{\alpha} \neq 0$. Prove that a vector field $\mathbb{X}$ tangent to S along α is parallel along α if and only if both $\|\mathbb{X}\|$ and the angle between $\mathbb{X}$ and α are constant along α.
14. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \rightarrow \mathbb{R}$ be a smooth function. Show that $\nabla_{e_i} f = (\partial f / \partial x_i)(p)$ where $p \in U$ and $e_i = (p, 0\ \ldots, 1, \ldots, 0)$.
15. Find the curvature κ of the plane curve $f^{-1}(c)$, oriented by $\nabla f / \|\nabla f\|$ where $f(x_1, x_2) = ax_1 + bx_2, \quad (a, b) \neq (0, 0)$.
16. Let V be a finite dimensional vector space with dot product and let $L : V \rightarrow V$ be a self-adjoint linear transformation on V. Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \rightarrow \mathbb{R}$ by $f(v) = L(v) \cdot v$. Suppose$f$ is staionary at $v_0 \in S$. Prove that $L(v_0) = f(v_0)v_0$.

(5 x 4 = 20)

PART C

Answer any 4 (10 marks each)

17.1. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \rightarrow \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

OR

2. Consider the vector field $\mathbb{X}(x_1, x_2) = (x_1, x_2, x_2, x_1)$ on $\mathbb{R}^2$. For $t \in \mathbb{R}$ and $p \in \mathbb{R}^2$, let $\varphi_t(p) = \alpha_p(t)$ where α_p is the maximal integral curve of $\mathbb{X}$ through p. Prove that $t \mapsto \varphi_t$ is a homomorphism from the additive group of real numbers into the group of one to one transformations of the plane.

18.1. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$, and let α be a piecewise smooth parametrized curve from p to q. Prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

OR

2. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0 \ \ \forall p \in S$. Prove that the Gauss map maps S onto the unit sphere S^n.

19.1. Prove that the Weingarten map Lp is self-adjoint.

OR

2. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by $\eta = -\dfrac{x_2}{x_1^2 + x_2^2}dx_1 + \dfrac{x_1}{x_1^2 + x_2^2}dx_2$. Prove that for $\alpha : [a, b] \to \mathbb{R}^2 - \{0\}$ any closed piecewise smooth parameterized curve in $\mathbb{R}^2 - \{0\}$, $\displaystyle\int_\alpha \eta = 2\pi k$ for some integer k.

20.1. (i) Find the Gaussian curvature of $\phi(t, \theta) = (\cos\theta, \sin\theta, t)$
(ii) Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite.

OR

2. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $\mathbf{v}$ be a unit vector in $S_p,\ p \in S$. Then prove that
(i) There exists an open set $V \subset \mathbb{R}^{n+1}$ containing p such that $S \cap \mathcal{N}(\mathbf{v}) \cap V$ is a plane curve.
(ii) The curvature at p of this curve is equal to the normal curvature $k(v)$.

(10 x 4 = 40)