Reg. No .... Name **21P2055**

M. Sc DEGREE END SEMESTER EXAMINATION - JULY 2021

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT10 : REAL ANALYSIS**

(For Regular - 2020 Admission & Supplementary - 2019/2018/2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

PART A
Answer All (1.5 marks each)

1. Show by an example that a function of bounded variation need not be continuous.
2. Prove that boundedness is a necessary condition for the property of bounded variation.
3. Prove that if $f \in \mathscr{R}$, then $f^2 \in \mathscr{R}$. Is converse true? Justify.
4. If $f_1, f_2 \in \mathscr{R}(\alpha)$ on [a,b], then prove that $f_1 + f_2 \in \mathscr{R}(\alpha)$ on $[a, b]$.
5. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin R$ on $[a, b]$ for any $a < b$.
6. Show by an example that there is a sequence of continuous functions, whose limit is discontinuous.
7. Find the derivative of the function $\int_0^{\sqrt{x}} e^{-t^2} dt$ at $x = 1$.
8. Discuss the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+1}$
9. Let $\{a_{ij}\}, i = 1, 2, 3, \ldots; j = 1, 2, 3, \ldots$ be a double sequence. If $\sum_{j=1}^{\infty} |a_{ij}| = b_i, i = 1, 2, 3, \ldots$ and $\sum b_i$ converges, prove that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$
10. Prove that $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$.

(1.5 x 10 = 15)

PART B
Answer any 4 (5 marks each)

11. Let **f**:[a,b]$\to \mathbb{R}^n$ be a path, where **f**=$(f_1, f_2 \ldots, f_n)$. Prove that $\bar{f}$ is rectifiable if and only if each component f_k is of bounded variation on [a,b].
12. Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\left[\frac{n}{2}\right]} \cos\left(\frac{k\pi}{n}\right)$
13. Prove that f is Riemann Stieltjes integrable on [a,b] iff for $\epsilon > 0$, there exist partition P of [a,b] such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.
14. Discuss the uniform convergence of the sequence $\{f_n\}$, where $f_n(x) = \frac{x^n}{1+x^n}$; $x \in [0, 1]$
15. State and prove Cauchy criterion for uniform convergence of a series of functions.
16. If z is a complex number with $|z| = 1$, prove that there is a unique $t \in [0, 2\pi)$ such that $E(it) = z$.

(5 x 4 = 20)

PART C

Answer any 4 (10 marks each)

17.1. Define equivalent paths. State and prove a necessary and sufficient condition for equivalence of two paths which are one to one on its domain. Give an example for non-equivalent paths.

OR

2. Prove that the graph of $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not rectifiable on [0,1].

18.1. Assume α increases monotonically and $\alpha^{'} \in \mathscr{R}$ on [a,b]. Let f be a bounded real function on [a,b]. Prove that $f \in \mathscr{R}(\alpha) \iff f\alpha^{'} \in \mathscr{R}$ and that $\int_a^b f d\alpha = \int_a^b f(x)\alpha^{'}(x)dx.$

OR

2. If f is continuous on [0,1] and if $\int_0^1 f(x)x^n dx = 0, n = 1, 2, 3, \ldots$, then prove that $f(x) = 0$ on [0,1].

a. State and prove the fundamental theorem of calculus.
b. Suppose that f is a real, continuously differentiable function on [a,b], $f(a) = f(b) = 0$ and $\int_a^b f^2(x)dx = 1$. Then prove that $\int_a^b xf(x)f^{'}(x)dx = -\frac{1}{2}.$

19.1. Construct a real valued function which is nowhere differentiable and continuous everywhere on $\mathbb{R}$.

OR

2. Prove or disprove: uniform limit of a sequence of differentiable function is differentiable. State a sufficient condition for the uniform limit of a sequence of functions being differentiable.

20.1. By considering e^x as a series, prove that

a. e^x is continuous and differentiable
b. $e^{x+y} = e^x + e^y \quad \forall x, y$

OR

2. Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and define $f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (|x| < R)$, then prove that $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$, no matter which $\epsilon > 0$ is chosen and the function f is contiuous and differentiable in $(-R, R)$ with $f'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1}$.

(10 x 4 = 40)