Reg. No .... Name 21P2043

M. Sc DEGREE END SEMESTER EXAMINATION - JULY 2021

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT09 : FUNCTIONAL ANALYSIS**

(For Regular - 2020 Admission & Supplementary - 2019/2018/2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer All (1.5 marks each)

1. If $T : D(T) \to Y$ is a bounded linear operator (where $D(T) \subset X$ and X and Y are normed spaces), then prove that

$$x_n \to x \text{ implies } Tx_n \to Tx(x_n \in D(T))$$

2. Prove that the inner product is conjugate linear in the second argument.
3. State and prove the Parallelogram equality in an inner product space.
4. If $x \neq 0, y \neq 0$ and $x \perp y$, then prove that $\{x, y\}$ is linearly independent.
5. Define the orthogonal complement of a closed subspace Y of a Hilbert space H.
6. If Y is a closed subspace of a Hilbert space H, prove that $Y \cap Y^{\perp} = \{0\}$.
7. If M is a non-empty subset of an inner product space, prove that $M \subset M^{\perp\perp}$.
8. If H is a Hilbert space, prove that $M^{\perp} = \{0\}$ implies M is total in H.
9. Prove that R under usual \`$\leq$' is a totally ordered set.
10. Show that a sub linear functional p on a vector space X satisfies

$$p(0) = 0 \text{ and } p(-x) \geq -p(x)$$

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. Prove that $\{x_1, x_2, \dots, x_n\}$, where $x_j(t) = t^j$, is a linearly independent set in $C[a, b]$.
12. Show that the operator $T : l^{\infty} \to l^{\infty}$ be defined by

$$y = (\eta_j) = Tx : \eta_j = \frac{\xi_j}{j}, x = (\xi_j),$$

is linear and bounded.

13. a. Define the algebraic reflexivity of a vector space
    b. Prove that a finite dimensional vector space is algebraically reflexive.
14. Let Y be a subspace of a Hilbert space H. Then prove that Y is complete if and only if Y is closed in H.
15. If a Hilbert space H contains a total orthonormal sequence, then prove that H is separable.
16. Prove that, if $x \in X$ and X is a normed space, then $||x|| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{||f||}$. Hence prove that, if $x_0 \in X$ is such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$

(5 x 4 = 20)

PART C

Answer any 4 (10 marks each)

17.1. a. State and prove Riesz's Lemma.
    b. In a finite dimensional normed space X, prove that any set $M \subset X$ is compact if and only if M is closed and bounded.

OR

2. a. Prove that dot product with a fixed vector $a \in R^3$ is a linear functional on R^3. Also prove that cross product with `a' is a linear operator on R^3. Will it be bounded? Justify.

b. Let $T : X \to Y$ be a linear operator. Show that the image of a subspace V of X is a vector space. Also prove that if $T^{-1}: Y\rightarrow X$ exists, then the inverse image of a subspace W of Y is a vector space.

c. Prove that a continuous mapping T of a compact subset M of a metric space X into R assumes a maximum and a minimum value at some points of M.

18.1. a. If Y is a Banach space and X is a normed space, prove that $B(X, Y)$ is a Banach space. Hence, prove that the dual space X' of a normed space X is a Banach space.

b. Prove that the dual space of R^n is R^n.

OR

2. a. Let Y be a subspace of a Hilbert space H. Then prove that:

(i) If Y is finite dimensional, then Y is complete.

(ii) If H is separable, so is Y.

b. Show that for a sequence (x_n) is an inner product space X the conditions

$$\|x_n\| \to \|x\| \text{ and } \langle x_n, x\rangle \to \langle x, x\rangle$$

imply the convergence $x_n \to x$.

19.1. a. An orthogonal set M in a Hilbert space H is total in H if and only if for all $x \in H$, the parseval relation holds.

b. Prove that two Hilbert spaces H and $\tilde{H}$ are isomorphic if and only if they have the same Hilbert dimension.

OR

2. a. Let $T : H \to H$ be a bijective bounded linear operator on a Hilbert space H. If T^{-1} is also bounded , prove that $(T^*)^{-1}$ exists and

$$(T^*)^{-1} = (T^{-1})^*$$

b. Prove that a linear operator T on a complex Hilbert space H is unitary if and only if T is isometric and subjective.

20.1. a. Define a sub linear functional on a real vector space X

b. State and prove Hahn Banach theorem for a real vector space.

OR

2. a. Let Y be a proper closed subspace of a normed space X. Let $x_0 \in X - Y$ and $\delta = \inf_{\tilde{y}\in Y} \|\tilde{y} - x_0\|$. Then prove that there exists an $\tilde{f} \in X'$ such that $\|\tilde{f}\| = 1, \tilde{f}(y) = 0$ for all $y \in Y$ and $\tilde{f}(x_0) = \delta$.

b. Under the same assumptions as above, prove that there exists a bounded linear functional h on X such that

$$\|h\| = \frac{1}{\delta}, h(y) = 0 \text{ for all } y \in Y$$

and $h(x_0) = 1$.

(10 x 4 = 40)