Reg. No .... Name **21P2003**

M Sc DEGREE END SEMESTER EXAMINATION - JULY 2021

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT06 : ABSTRACT ALGEBRA**

(For Regular - 2020 Admission and Supplementary 2019/2018/2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer All (1.5 marks each)

1. How many elements of finite order does $\mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_4$ have?
2. What are the possible numbers of Sylow 2-subgroups of a group of order 24?
3. Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$ of order 4.
4. Give an example of an infinite torsion group.
5. Find the sum and product of $f(x) = 4x - 5$ and $g(x) = 2x^2 - 4x + 2$ in $\mathbb{Z}_8[x]$.
6. Find all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[x]/ < x^3 + x^2 + c >$ is a field?
7. Are i and $-i$ conjugate over $\mathbb{R}$? Over $\mathbb{C}$? Justify your answer.
8. True or False: There are no proper fields lying between $\mathbb{Q}$ and $\mathbb{R}$.Justify your answer.
9. True or False: $\mathbb{R}$ is a splitting field over $\mathbb{Q}$.Justify.
10. True or False: $\mathbb{R}$ is not perfect. Justify your answer.

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. (a) State the fundamental theorem of Finitely generated abelian groups.
(b) Find all abelian groups upto isomorphism of order 720.
12. Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.
13. Define algebraic and transcendental numbers. Show that the set of all algebraic numbers forms a field.
14. Let E be an extension field of F and $\alpha \in E$ be algebraic over F.Establish the existence and uniqueness of the irreducible polynomial for α over F.
15. Let F be a finite field of characteristic p.Show that the map $\sigma_p : F \to F$ defined by $\sigma_p(a) = a^p$ for $a \in F$ is an automorphism.What is this automorphism called? What is the fixed field of σ_p?
16. Show that if $E \leq \overline{F}$, every irreducible polynomial in $F[x]$ having a zero in E splits in E.

(5 x 4 = 20)

PART C

Answer any 4 (10 marks each)

17.1. (a). Let X be a G-set and let $x \in X$. Under the usual notations, Show that $|Gx| = (G : G_x)$.Further show that if $|G|$ is finite, then $|Gx|$ is divisor of $|G|$.
(b). Let H be a p-subgroup of a finite group G.
Show that $(N[H] : H) \equiv (G : H)(mod\, p)$.
Further show that if p divides $(G : H)$, show that $N[H] \neq H$.

OR

2. (a). Find the conjugate classes and the class equation for S_4.
(b). Find the class equation of S_5.

18.1. (a). Show that a nonzero polynomial $f(x) \in F[x]$ of degree n has at most n zeroes in a field F

(c). Show that the multiplicative group of all nonzero elements of a finite field is cyclic.

OR

2. (a). State and prove Kronecker's Theorem.

(b). Construct a field containing four elements.

19.1. Stating the necessary lemmas, establish the existence and uniqueness of $\mathbf{GF}(p^n)$, the Galois field of order p^n.

OR

2. State and prove the Conjugation Isomorphisms Theorem for field theory.

20.1. State the main theorem of Galois Theory and illustrate each of the properties using a specific example.

OR

2. Find the splitting field K of $x^4 + 1$ over $\mathbb{Q}$. Compute $G(K/\mathbb{Q})$, find its subgroups and the corresponding fixed fields and draw the subgroup and subfield lattice diagrams.

(10 x 4 = 40)