Reg. No.. Name.. **20P1003**

M. Sc. DEGREE END SEMESTER EXAMINATION - OCT 2020: FEBRUARY 2021

SEMESTER – 1: MATHEMATICS

COURSE: 16P1MATT01: LINEAR ALGEBRA

(Common for Regular-2020 Admission & Supplementary 2019/2018/2017/2016 Admissions)

Time: Three Hours Max. Marks: 75

SECTION A

Answer All (1.5 marks each)

1. Let V be a vector space over the field F. Show that the intersection of any collection subspaces of V is a subspace of V .
2. Find a basis for the space of all 2 x 2 matrices with complex entries satisfying $A_{11} + A_{22} = 0$.
3. Prove that the set $S = \{\alpha + i\beta, \gamma + i\delta\}$ is a basis for the vector space C over R if and only if and only if $\alpha\delta - \beta\gamma \neq 0$.
4. Is there a linear transformation T from R^3 into R^2 such that $T(1,-1,1) = (1,0)$ and $T(1,1,1) = (0,1)$? Justify.
5. Let $\mathbb{R}$ be the field of real numbers and let V be the space of all functions from $\mathbb{R}$ into $\mathbb{R}$ which are continuous. Define T by $(Tf)(x) = \int_0^x f(t)dt$.Show that T is a linear transformation from V into V.
6. Define a non-singular transformation. Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, y)$ is non-singular.
7. Define commutative and non-commutative rings. Give examples for each.
8. Let E be a projection on V with range R and null space N.Show that $V = R \oplus N$.
9. Show that similar matrices have the same characteristic polynomial.
10. Define invariant subspace with an example. Also state a necessary condition for a subspace to be invariant. (1.5 x 10 = 15)

SECTION B

Answer any 4(5 marks each)

11. Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \dots \dots \beta_m$. Show that any independent set of vectors in V is finite and contains no more than m elements.
12. Let A be an $n \times n$ matrix over a field F and suppose that the row vectors of A form a linearly independent set of vectors in F^n. Show that A is invertible.
13. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (-y, x)$.

    i) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

    ii) What is the matrix of T in the ordered basis $B = \{(1,2), (1, -1)\}$?

14. Show that {(1, 2),(3, 4)} is a basis for $\mathbb{R}^2$. Let T be the unique linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $T(1, 2)$ = (3, 2, 1) and T (3, 4) = (6, 5, 4). Find $T(1,0)$.
15. Let A be an $n \times n$ matrix with λ as an eigen value. Show that,
    a) $k + \lambda$ is an eigen value of $A + kI$.
    b) If A is non-singular, $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

16. Find the characteristic values and characteristic vectors of the matrix A = $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

(5 X 4 = 20)

SECTION C

Answer any 4(10 marks each)

17 1. Let V be an n-dimensional vector space over the field F and let $\mathscr{B}$ and $\mathscr{B}'$ be two ordered bases of V. Show that there is a unique necessarily invertible $n \times n$ matrix P with entries in F such that $[\alpha]_{\mathscr{B}} = P[\alpha]_{\mathscr{B}'}$ and $[\alpha]_{\mathscr{B}'} = P^{-1}[\alpha]_{\mathscr{B}}$.

OR

2.a) Let W be the set of all (x_1, x_2 , x_3, x_4, x_5) in R^2 which satisfy

$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$

$x_1 + \frac{2}{3}x_3 - x_5 = 0$

$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$. Find a finite set of vectors which spans W

b) Let R be a non-zero row reduced echelon matrix. Prove that the non-zero vectors of R form a basis for the row space of R.

18. 1. (a) Define rank and nullity of a linear transformation.

(b) Let V be finite dimensional and $T : V \to W$ be a linear transformation. Prove that $rank\ (T) + nullity(T) = \dim V$.

(c) Determine a linear transformation from R^3 into R^3 which has its range the subspace spanned by (1 ,0 ,1) and (1, 2, 2). What is Nullity of such a linear transformation?

OR

2. (a) Does there exist a linear transformation $T: R^3 - R^2$ such that

$T(1, -1, 1) = (1,0)$ and $T(1,1,1) = (0,1)$? Justify.

(b) Let V and W be finite-dimensional vector spaces over the field F.Prove that V and W are isomorphic if and only if $\dim V$ = dim W.

(c) Let T be the linear operator on R^2 defined by $T(x_1, x_2) = (x_1, 0)$. computer the matrix of T relative to the ordered basis {(1, 1), (2, 1)}.

19. 1. (a)Let D be a n-linear function on the space of $n \times n$ matrices over a field K. Suppose D has the property that D(A) =0 whenever two adjacent rows of A are equal. Show that D alternating.

(b) Let $n > 1$ and let D be an alternating $(n-1)$ linear function on an $(n-1) \times (n-1)$ matrix over K. Show that for each $j, j = 1, \dots\dots, n,$ the function E_j defined by $E_j(A)$=$\sum_{i=1}^{n} = (-1)^{(i+j)} A_{ij} D_{ij}(A)$ is an alternating n-linear function on the space of $n \times n$ matrix A . If D is the determinant function, so is E_j.

OR

2.(a) If A is an $n \times n$ skew symmetric matrix with complex entries and n is odd, prove that det A = 0.

(b)If A is an $n \times n$ invertible matrix over a field F, show that det $A \neq 0$.

20. 1. (a) Let T be a diagonalizable linear operator on a space V.

If $c_1, \dots\dots, c_k$ are the distinct characteristic values of T, prove that the minimal polynomial for T is $(x - c_1), (x - c_2) \dots\dots (x - c_k)$.

(b) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V . Show that T is triangulable if and only if the minimal polynomial of T is a product of linear polynomials over F .

OR

2. (a) Let T be a linear operator on a finite dimensional space V. Let $c_1, c_2, \dots\dots c_k$ be the distinct characteristic values and $W_1, W_2, \dots\dots, W_k$ be the corresponding characteristic spaces. Prove that $dim(W_1 + W_2 + \dots\dots + W_k)$ = $\dim W_1 + dim\, W_2 + \dots + dim W_k$.

(b)If W_1 and $W_2$are subspaces of V then prove that they are independent if and only if $W_1 \cap W_2 = 0$.

(10 x 4 = 40)

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