20P4040

Reg. No .... Name ....

MSc DEGREE END SEMESTER EXAMINATION - MARCH 2020

SEMESTER 4 : MATHEMATICS

COURSE : **16P4MATT19EL : THEORY OF WAVELETS**

(For Regular - 2018 Admission and Supplementary 2017, 2016 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer All the Following (1.5 marks each)

1. If $z=(1,2,0,4)$, find $z(9), z(-2)$ and $z(0)$.
2. If $w \in l^2(Z_N)$, prove that $\check{w}(n+N)=\check{w}(n)$ for all $n \in Z_N$.
3. When we say there is a perfect reconstruction in the filter bank?
4. If N is divisible by 2^l, define $U^l : l^2(Z^l_{N/2})->l^2(Z_N)$.
5. Define a p^{th} stage wavelet basis for $l^2(Z_N)$ is N is divisible by 2^p (p is a positive integer).
6. If $\psi_{-j,k}=R_{2^jk}f_i$ and $\phi_{-j,k}=R_{2^jk}g_j$, prove that $\hat{\phi}_{-j,k}(m)=e^{-2\pi i2^jkm/N}\hat{\psi}_{-j,0}(m)$ and $\hat{\phi}_{-j,k}(m)=e^{-2\pi i2^jkm/N}\ \hat{\phi}_{-j,0}(m)$.
7. Which of the following sequences is square summable?
   (i) $(Z(n))_{n=1}^{\infty}$, where $z(n)=\frac{1}{\sqrt{n}}$
   (ii)$(w(n))_{n=1}^{\infty}$, where $w(n)=\frac{1}{n}$
8. Suppose that $\{z_k\}_{k=M}^{\infty}$ is a cauchy sequence in $l^2(Z)$. Prove that $\{z_k(n)\}_{k=M}^{\infty}$ is a cauchy sequence in C for all $n \in Z$.
9. If $z \in l^2(Z)$, prove that $(z^*)^\wedge(\theta)=\hat{z}(\theta+\pi)$
10. If $z, w \in l^2(Z)$, prove that $<R_jz, R_kw>=<z, R_{k-j}, w>$ for all $k, j \in Z$.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. (a) Define convolution in $l^2(Z_N)$
    (b) Is it a binary operation in $l^2(Z_N)$? Is it commutative?
    (c) If z=(1,1,0,2) and $w=(i,0,1,i) \in l^2(Z_4)$, find $z*w$
12. Suppose N is even and N=2M. Let $z \in l^2(Z_N)$ and $x, y, w \in l^2(Z_{N/2})$. Then prove that $D(z)*w=D(z*U(w)$ and $U(x)*U(y)=U(x*y)$.
13. Suppose N is divisible by 2 and $u_1 \in l^2(Z_N)$
    Define $u_2 \in l^2(Z_{N/2})$ by $u_2(n)=u_1(n)+u_1(n+\frac{N}{2})$ Then prove that $\hat{u_2}(m)=\hat{u_1}(2m)$.
14. i) With the usual notations, prove that S_A is a subspace of H, if A= $\{a_j\}_{j\in Z}$ is an orthonormal set in a Hilbert space H .
    ii) If $z \in l^2(Z)$ and $<z, a_j>=0$ for all $j \in Z$, prove that $<z, s>=0$ for all $s \in S_A$.
15. Suppose $\theta_0 \in (-\pi,\pi)$ and $\alpha>0$ is sufficiently small that $-\pi<\theta_0-\alpha<\theta_0+\alpha<\pi$ .Define $I=(\theta_0-\alpha, \theta_0+\alpha)$ and J=$(\theta_0-\frac{\alpha}{2}, \theta_0+\frac{\alpha}{2})$ Then prove that there exists a $\delta>0$ and a sequence of real valued trigonometric polynomials $\{p_n(\theta)\}_{n=1}^{\infty}$ such that
    (i) $p_n(\theta)\geq 1$ for $\theta \in I$.
    (ii) $p_n(\theta)\geq(1+\delta)^n$ for $\theta \in J$.

(iii) $|p_n(\theta)| \leq 1$ for $\theta \in [-\pi, \pi) - I$.

16. (i) Prove that $l^1(Z)$ is a normed space.
(ii) If $z, w \in l^1(Z)$, prove that $z * w \in l^1(Z)$.

(5 x 4 = 20)

Section C
Answer any 4 (10 marks each)

17.1. (a) Define the IDFT $\check{w}$ of $w \in l^2(Z_N)$
(b) Prove that $(\hat{z})^{\vee} = z$ and $(\check{w})^{\wedge} = w$ for all $z, w \in l^2(Z_N)$
(c) Prove that $'\wedge'$ is both one-one and onto.

OR

2. (a) Suppose N is even and N=2M for some $M \in N$. If $w \in l^2(Z_N)$, then prove that $\{R_{2k}w\}_{k=0}^{M-1}$ is an orthonormal set with M elements if and only if $|\hat{w}(n)^2| + |\hat{w}(n+M|^2 = 2$ for all n=0,1,2,..,M-1.
(b) In the above statement prove that the phrase "With M elements" is significant.

18.1. (a) Define p^{th} stage wavelet basis for $l^2(Z_N)$ if N is divisible by 2^p.
(b) Suppose N is divisible by 2^l, $g_{l-1} \in l^2(Z_N)$ and the set $\{R_{2^{l-1}k}\, g_{l-1}\}_{k=0}^{\frac{N}{2^{l-1}}-1}$ is orthonormal with $N/2^{l-1}$ elements. Suppose $u_l, v_l \in l^2(Z_{N/2}^{l-1})$ and the system matrix $A_l(n)$ is unitary for all $n = 0, 1, 2, .., (N/2^l) - 1$.
Define $f_l = g_{l-1} * U^{l-1}(v_l)$ and $g_l = g_{l-1} * U^{l-1}(u_l)$
Then prove that $\{R_{2^l k} f_l\}_{k=0}^{\frac{N}{2^l}-1} \bigcup \{R_{2^l k} g_l\}_{k=0}^{\frac{N}{2^l}-1}$ is an orthonormal set with $N/2^{l-1}$ elements.

OR

2. Describe Daubechie's D_6 wavelet system on Z_N.

19.1. Let $\{a_j\}_{j \in Z}$ be an orthonormal set in a Hilbert space H. Then Prove that the following statements are equivalent
i) $\{a_j\}_{j \in Z}$ is Complete
ii) For any f,g $\in H$, $< f, g >= \sum_{j \in Z} < f, a_j > \overline{< g, a_j >}$
iii) For any f $\in H$, $||f||^2 = \sum_{j \in Z} | < f, a_j > |^2$.

OR

2. i) If $z = (z(n))_{n \in Z} \in l^2(Z)$, prove that $\sum_{n \in Z} z(n) e^{in\theta}$ converges to an element of $L^2([-\pi, \pi))$.
ii) State and prove Plancherel's formula in $L^2([-\pi, \pi))$.
ii) State and prove Parseval's relation in $L^2([-\pi, \pi))$.
iii) State and prove Fourier inversion formula in $L^2([-\pi, \pi))$.

20.1. Suppose $w, z \in l^1(Z)$. Then prove that
(i)$\{R_{2k}w\}_{k \in Z}$ is orthonormal if and only if $|\hat{w}(\theta)|^2 + |\hat{w}(\theta + \pi)|^2 = 2$ for all $\theta \in [0, \pi)$ and
(ii) $< R_{2k}z, R_{2j}w >= 0$ for all $k, j \in Z$ if and only if $\hat{z}(\theta)\overline{\hat{w}(\theta)} + \hat{z}(\theta + \pi)\overline{\hat{w}(\theta + \pi)} = 0$ for all $\theta \in [0, \pi)$.

OR

2. With the usual notations prove that $V_{-l} \oplus W_{-l} = V_{-l+1}$.

(10 x 4 = 40)