Reg. No .... Name ....

MSc DEGREE END SEMESTER EXAMINATION - MARCH 2020

SEMESTER 4 : MATHEMATICS

COURSE : **16P4MATT16EL : DIFFERENTIAL GEOMETRY**

(For Regular - 2018 Admission and Supplementary - 2017, 2016 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer All the Following (1.5 Marks each)

1. Describe the level sets of $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$, for $n = 0, 1, 2$.
2. Define gradient of a scalar function.
3. Sketch the vector field on $\mathbb{R}^2$: $\mathbb{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (-x_1, -x_2)$.
4. Find the velocity, the acceleration, and the speed of parametrized curve $\alpha(t) = (\cos t, \sin t, t)$.
5. Define covariant derivative of a parallel vector field.
6. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.
7. Prove that the value of $\nabla_v(f)$ is independent of the curve α in S passing through p with velocity v.
8. Find the length of the parameterized curve $\alpha(t) = (t^2, t^3),\ I = [0, 2]$
9. Let $\alpha : I \to \mathbb{R}^2$ defined by $\alpha(t) = (\gamma \cos t, \gamma \sin t)$. Find $l(\alpha)$.
10. State inverse function theorem for n-surface.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 Marks each)

11. Show by an example that the set of vectors tangent at a point p of a level set might be all of $\mathbb{R}_p^{n+1}$.
12. Determine whether the vector field $\mathbb{X}(x_1, x_2) = (x_1, x_2, 1 + x_1^2, 0)$ where $U = \mathbb{R}^2$ is complete or not.
13. Show that a parametrized curve α in the unit n-sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ is a geodesic if and only if it is of the form $\alpha(t) = e_1 \cos at + e_2 \sin at$ for some orthogonal pair of unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$.
14. Find the length of the parameterized curve $\alpha(t) = (\cos t, \sin t, \cos t, \sin t),\ I = [0, 2\pi]$
15. Compute $\nabla_v f$ where $f(q) = q \cdot q$, $\mathbf{v} = (p, v)$.
16. Let S be an n-surface in $\mathbb{R}^{n+1}$ and $p \in S$. Show that the subset of S consisting of all points $q \in S$ which can be joined to p by a continuous curve in S is a connected n-surface.

(5 x 4 = 20)

Section C
Answer any 4 (10 Marks each)

17.1. Let $\mathbb{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Prove that there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ of $\mathbb{X}$ such that
(i) $\alpha(0) = p$
(ii) If $\beta : \tilde{I} \to U$ is any other integral curve of $\mathbb{X}$ with $\beta(0) = p$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in I$

OR

2. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\mathbb{X}$ be a smooth tangent vector field on S, and let $p \in S$. Prove that there exists an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that
(i) $\alpha(0) = p$
(ii) $\dot{\alpha}(t) = \mathbb{X}(\alpha(t))$ for all $t \in I$
(iii) If $\beta : \tilde{I} \to S$ is any other parametrized curve in S satisfying (i) and (ii), then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

18.1. Show that if the spherical image of a connected n-surface S is a single point then S is part or all of an n-plane.

OR

2. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$, and let $v \in S_{\alpha(t_0)}$. Show that there exists a unique vector field $\mathbb{V}$, tangent to S along α which is parallel and has $\mathbb{V}(t_0) = v$.

19.1. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by $\eta = -\dfrac{x_2}{x_1^2 + x_2^2} dx_1 + \dfrac{x_1}{x_1^2 + x_2^2} dx_2$. Prove that for $\alpha : [a, b] \to \mathbb{R}^2 - \{0\}$ any closed piecewise smooth parameterized curve in $\mathbb{R}^2 - \{0\}$, $\displaystyle\int_\alpha \eta = 2\pi k$ for some integer k.

OR

2. Define local parametrization of a plane curve C and prove that the local parametrization is unique upto a reparametrization.

20.1. Find the Gaussian curvature for
(i) $\psi(\theta, \phi) = ((a + b\cos\phi)\cos\theta), (a + b\cos\phi)\sin\theta), \sin\phi)$.
(ii) $x_1^2 + x_2^2 - x_3^2 = 0, \; x_3 > 0$

OR

2. Let $\varphi : U \to \mathbb{R}^{n+1}$ be a parametrized n-surface in $\mathbb{R}^{n+1}$ and let $p \in U$. Prove that there exists an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$.

(10 x 4 = 40)