20P2043

Reg. No .... Name ....

M. Sc DEGREE END SEMESTER EXAMINATION - MARCH 2020

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT09 : FUNCTIONAL ANALYSIS**

(For Regular - 2019 Admission & Supplementary 2018/2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer All the Following (1.5 marks each)

1. Define equivalent norms. Give two equivalent norms in R^2.
2. Prove that the dot product with a fixed vector $a \in R^3$ is a bounded linear functional defined on R^3 with norm $||a||$.
3. Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$, prove that $x_0 = 0$
4. If X is a real inner product space, show that $||x|| = ||y||$ implies $\langle x + y, x - y \rangle = 0$
5. Define the direct sum of two subspaces Y and Z of a vector space X.
6. If Y is a closed subspace of a Hilbert space H, prove that $Y^\perp$ is also a closed subspace of H.
7. If M is a non-empty subset of an inner product space X, prove that $M^\perp \neq \phi$
8. Define total orthonormal set in an inner product space X.
9. Let X be a normed space and X' its dual space. If $X \neq \{0\}$, show that $X' \neq \{0\}$
10. If p is a sub linear functional defined on a vector space X, prove that $|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each) **(5 x 4 = 20)**

11. Show that in an n-dimensional vector space X, the representation of any x as a linear combination of a given basis vectors, $e_1, e_2, \ldots, e_n$ is unique.
12. Prove that the range of a bounded linear operator need not be closed even though the null space of the operator is closed.
13. Prove that in an inner product space X. $x \perp y$ if and only if $\|x + \alpha y\| = \|x - \alpha y\|$ for all scalars α.
14. Prove that in a complex inner product space X

$$Re\langle x, y \rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2]$$

and $Im\langle x, y \rangle = \frac{1}{4}\left[\|x + iy\|^2 - \|x - iy\|^2\right]$.

15. If S and T are normal operators satisfying $ST^* = T^*S$ and $TS^* = S^*T$, show that $S + T$ and ST are normal.
16. If X and Y are Banach spaces and $T_n \in B(X, Y)$; $n = 1, 2, 3, \ldots$, show that the following statements are equivalent.

   a. $(||T_n||)$ is bounded
   b. $(||T_n x||)$ is bounded for all $x \in X$
   c. $(|g(T_n x)|)$ is bounded for all $x \in X$ and all $g \in Y'$

Section C
Answer any 4 (10 marks each)

17.1. a. Define convergence and absolute convergence of an infinite series in a normed space.
b. Show that in a Banach space absolute convergence always imply convergence
c. Show that in a normed space absolute convergence need not imply convergence.

OR

2. a. Suppose c is the set of all convergent sequences of scalars
(i) Prove that $c \subset l^\infty$.
(ii) Prove that c is a subspace of l^∞
(iii) Prove that c is a closed subspace of l^∞
b. Give an example of a non-closed subspace of l^∞
c. Prove that a linear operator preserves linear dependence.

18.1. a. Define the dual basis of a basis for an n dimensional vector space. hence, prove that, if X is an n-dimensional vector space, then $\dim X^* = \dim X^{**} = n$
b. Prove that the dual space of l^1 is l^∞

OR

2. a. Let Y be a subspace of a Hilbert space H. Then prove that Y is complete if and only if Y is closed in H.
b. Prove that l^p with $p \neq 2$ is not an inner product space. Is l^2 an inner product space? Justify.
c. Let $T : X \to X$ be a bounded linear operator on a complex inner product space X. If $< Tx, x >= 0$ for all $x \in X$, show that $T = 0$.

19.1. a. State and prove Bessel inequality
b. Explain the Gram-schmidt process for orthonormalizing a linearly independent set.
c. Let X be the inner product space of all real valued continuous functions defined on $[-1, 1]$ with inner product $\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)dt$. Then orthonormalize the first three terms of the sequence $(x_0, x_1, x_2, \ldots,)$, where $x_j(t) = t^j$

OR

2. a. State and prove Riesz's theorem(Functionals on Hilbert spaces).
b. Prove that , for any fixed $z \in H$, the functional f defined on H by $f(x) = \langle x, z\rangle$ for all $x \in H$, is bounded linear and $\|f\| = \|z\|$.

20.1. a. Define the adjoint operator $T^\times$ of a bounded linear operator $T : X \to Y$, where X and Y are normed spaces
b. Prove that $T^\times$ is bounded linear and $\|T^\times\| = \|T\|$
c. If $S, T \in B(X, Y)$, prove that $(S + T)^\times = S^\times + T^\times$.

OR

2. a. State Baire's category theorem
b. State and prove uniform boundedness theorem.

(10 x 4 = 40)