Reg. No .... Name ....

MSc DEGREE END SEMESTER EXAMINATION - MARCH 2020

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT06 : ABSTRACT ALGEBRA**

(For Regular - 2019 Admission and Supplementary - 2018/2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer All the Following (1.5 marks each)

1. What are the possible numbers of Sylow 5-subgroups of a group of order 255?
2. Does every abelian group of order divisible by 4 contain a cyclic subgroup of order 4? Justify your answer.
3. Is a direct product of cyclic groups cyclic? Justify your answer.
4. Give an example of an infinite finitely generated abelian group.
5. Is $\mathbb{Q}[x]\Big/\langle x^2+6x+6\rangle$ a field? Justify your answer.
6. Show that $R[x]/\langle x^2+1\rangle \cong \mathbb{C}$.
7. Define primitive n^{th} root of unity in a field.Give an example.
8. Define the Frobenius automorphism σ_p? What is $F_{\{\sigma_p\}}$?
9. What is the order of $G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$?
10. True or False: $\mathbb{R}$ is a splitting field over $\mathbb{R}$.Justify.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. (a). Find the decomposition of D_4 into conjugacy classes.
    (b). Write the class equation for D_4.
12. Show that a group of order 96 is not simple.
13. Prove that if D is an integral domain, then show that $D[x]$ is an integral domain.
14. Consider the evaluation homomorphism $\phi_4 : \mathbb{Z}_7[x] \to \mathbb{Z}_7$. Evaluate $\phi_4(3x^{106}+5x^{99}+2x^{53})$.
15. Show that if γ is constructible and $\gamma \notin \mathbb{Q}$, then $[\mathbb{Q}(\gamma):\mathbb{Q}] = 2^r$, for some integer $r \geq 0$.
16. Show that if $[E:F]=2$, then E is a splitting field over F.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. (a). Find all Sylow 3-subgroups of S_4 and show that they are all conjugate.
(b). Find two Sylow 2-subgroups of S_4 and show that they are conjugate.
(c). Show that there are no simple groups of order $p^r m$, where p is a prime, r is a positive integer and $m < p$.

OR

2. (a) Let G be an abelian group of order 72.
   (i) Can you say how many subgroups of order 8 G has?
   (ii) Can you say how many subgroups of order 4 G has?

(b) Prove that every group of order $(35)^3$ has a normal subgroup of order 125?
(c) Prove that every group of prime power order is solvable.Is the converse true? Justify your answer.

18.1. (a). Let R be a ring, and let R^R be the set of all functions mapping R into R. For $\phi, \psi \in R^R$, define the sum $\phi + \psi$ and the product $\phi . \psi$ by

$$(\phi + \psi)(r) = \phi(r) + \psi(r)$$
$$(\phi . \psi)(r) = \phi(r)\psi(r)$$

for $r \in R$, respectively. Show that $< R^R, +, . >$ is a ring.
(b). How many elements are there in $\mathbb{Z}_2^{\mathbb{Z}_2}$ and $\mathbb{Z}_3^{\mathbb{Z}_3}$?

OR

2. (a). Show that a polynomial $f(x) \in F[x]$ of degree 2 or 3 is reducible over F if and only if it has a zero in F.Is the result true for polynomials of degree ≥ 4? Justify your answer.
(b). How is the reducibility of polynomials over $\mathbb{Z}$ related over their reducibility over $\mathbb{Q}$? Show that if $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{Z}[x]$ with $a_0 \neq 0$ and if $f(x)$ has a zero in $\mathbb{Q}$, then it has a zero $m \in \mathbb{Z}$ and m must divide a_0.
(c). Show that $f(x) = x^4 - 2x^2 + 8x + 1$ is irreducible over $\mathbb{Q}$.

19.1. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $F = \mathbb{Q}$.Show that K is a finite separable extension of F.

OR

2. (a). Let α be algebraic of degree n over F.Show that there are at most n different isomorphisms of $F(\alpha)$ onto a subfield of $\overline{F}$ and leaving F fixed.
(b). Describe all extensions of the identity map of $\mathbb{Q}$ to an isomorphism mapping $\mathbb{Q}(\sqrt[3]{2})$ onto a subfield of $\overline{\mathbb{Q}}$.

20.1. (a). State and prove *The Primitive Element Theorem*
(b). Show that a finite extension of a field of characteristic zero is a simple extension.

OR

2. Show that every finite field is perfect.

(10 x 4 = 40)