Reg. No .... Name **19P4040**

MSc DEGREE END SEMESTER EXAMINATION - MARCH/APRIL 2019

SEMESTER 4 : MATHEMATICS

COURSE : **16P4MATT19EL : THEORY OF WAVELETS**

(For Regular - 2017 Admission and Supplementary 2016 Admission)

Time : Three Hours Max. Marks: 75

Section A

Answer all the following (1.5 marks each)

1. Prove that $\hat{z}(m) = \sqrt{N} < z, E_m >; 0 \leq m \leq N-1$.
2. If $B = \{R_{2k}v\}_{k=0}^{M-1} \bigcup \{R_{2k}u\}_{k=0}^{M-1}$ is a first stage wavelet basis for $l^2(Z_N)$, then represent the construction of $[z]_B$ for any $z \in l^2(Z_N)$ by a filter bank diagram.
3. Describe the analysis phase and synthesis phase of a filter bank using diagram.
4. Prove that the sum of the number of components of all output vectors of the analysis phase of the p^{th} stage filter bank = N.
5. If N is divisible by 2^p, with the usual notations define f_l, g_l for $l = 1, 2, 3, \ldots., p$.
6. If $\psi_{-j,k} = R_{2^j k}\, f_j and\, \phi_{-j,k} = R_{2^j k}\, g_j$, prove that $\psi_{-j,0} = f_j and\, \phi_{-j,0} = g_j$.
7. Define the absolute convergence of $\sum\limits_{n \in Z} w(n)$.
8. When we say a complex valued function f defined on $[-\pi, \pi)$ is square integrable over $[-\pi, \pi)$?.
9. Define the down sampling operator $D : l^2(Z) \to l^2(Z)$ and the upsampling operator $U : l^2(Z) \to l^2(Z)$.
10. If $z = (z(n))_{n \in Z} \in l^2(Z)$, prove that $DoU(z) = z$.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. (a) If $z = (z(0), z(1), \ldots, z(N-1)) \in l^2(Z_N)$, what is $\overline{z}$?
    Prove that $(\overline{z})^\wedge(m) = \overline{\hat{z}(N-m)} : 0 \leq m \leq N-1$
12. Suppose $z, w \in l^2(Z_N)$. Prove that
    (i) $(z * w) = \tilde{z} * \tilde{w}$.
    (ii) $[D(z)]^\sim = D(\tilde{z})$, if N is even and
    (iii) $[U(z)]^\sim = U(\tilde{z})$
13. Suppose N is divisible by 2^p . Suppose $u_1, v_1, \in l^2(Z_N)$ are such that the system matrix A(n) of u_1 and v_1 is unitary for all n.
    Define $u_l(n) = \Sigma_{k=0}^{2^{l-1}-1} u_1(n + \frac{kN}{2^{l-1}})$ and $v_l(n) = \Sigma_{k=0}^{2^{l-1}-1} v_1(n + \frac{kN}{2^{l-1}})$ for

$l = 2, 3, \ldots, p$. If we define $f_1 = v_1$ and $g_1 = u_1$ and for $l = 2, 3, .., p$ if we define $f_l = g_{l-1} * U^{l-1}(v_l)$ and $g_l = g_{l-1} * U^{l-1}(u_l)$,

Then prove that $U_1, V_1, U_2, V_2 \ldots. U_p, V_p$ is a P^{th} stage wavelet filter sequence.

14. i) When we say $\{z_k\}_{k=M}^{\infty} (M \in Z)$ is a Cauchy sequence in $l^2(Z)$?
    ii) Prove that $l^2(Z)$ is Complete.

15. Suppose $u_1 v \in l^1(z)$ are such that $[R_{2k}v]_{k \in z} \cup [R_{2k}u]_{k \in z}$ is a first stage wavelet system for $l^2(z)$. Suppose also that $u(n) = v(n) = 0 \forall n < 0$ and $n > N - 1$. Define $u_{(N)}, v_{(N)} \in l^2(Z_N)$ by $u_{(N)}(n) = u(n)$ and $v_{(N)}(n) = v(n)$ for $n = 0, 1, \cdots N - 1$. Then prove that $[R_{2k}v_{(N)}]_{K=0}^{M-1} \cup [R_{2k}u_{(N)}]_{K=0}^{M-1}$ is a first stage wavelet basis for $l^2(z_n)$.

16. (i) Let $w \in l^2(Z)$. Then prove that $\{R_{2k}w\}_{k \in Z}$ is orthonormal if and only if $\langle w, R_{2k}w \rangle = 1$ if $k = 0$

    $\langle w, R_{2k}w \rangle = 0$ if $k \neq 0$

    for all $k \in Z$.

    (ii) Suppose $z, w \in l^2(Z)$. Prove that

    (a) $U(z * w) = U(z) * U(w)$.

    (b) $[U(z)]^{\sim} = U(\tilde{z})$.

    (c) $(z * w)^{\sim} = \tilde{z} * \tilde{w}$.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. (a) When we say a linear transformation $T : l^2(Z_N) \to l^2(Z_N)$ is translation invariant?

(b) Let $T : l^2(Z_N) \to l^2(Z_N)$ be a translation invariant linear transformation. Then prove that each element of the Fourier basis F is an eigen vector of T.

OR

2. (a) Suppose $z, w \in l^2(Z_N)$. Then prove that for each m,

    $(z * w)^{\wedge}(m) = \hat{z}(m)\, \hat{w}(m)$

    (b) Let z=(1,0,1,0) and w=(0,1,0,1) be two elements of $l^2(Z_4)$. Find $z * w$.

    (c) Find $(z * w)^{\wedge}$

    (d) Find $\hat{z}$ and $\hat{w}$ and verify $(z * w)^{\wedge}(m) = \hat{z}(m)\hat{w}(m)$ for m=0,1,2,3.

18.1. Suppose N is divisible by 2^p, $z \in l^2(Z_N)$ and $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ are such that $u_l, v_l, \in l^2(Z_{N/2}^{l-1})$ for l=1,2,...,p

(a) Define $x_1, x_2, \ldots., x_p, y_1, y_2, \ldots., y_p$

(b) Define $f_1, f_2, \ldots., f_p, g_1, g_2, \ldots., g_p$

(c) For $l = 1, 2, .., p$, prove that

$x_l = D^l(z * \tilde{f}_l)$ and $y_l = D^l(z * \tilde{g}_l)$

OR

2. Describe real shannon wavelet system.

19.1. .(i) Suppose H is a Hilbert Space, $\{a_j\}_{j\in Z}$ is an orthonormal set in H and $z \in l^2(Z)$. Then prove that the series $\sum\limits_{j\in Z} z(j)a_j$ converges in H and

$$||\sum_{j\in Z} z(j)a_j||^2 = \sum_{j\in Z} |z(j)|^2.$$

(ii). Suppose H is a Hilbert space, $\{a_j\}_{j\in Z}$ is an orthonormal set in H and $f \in H$. Then prove that $\{< f, a_j >\}_{j\in Z} \in l^2(Z)$ and $\sum\limits_{j\in Z} |< f, a_j >|^2 \leq ||f||^2$.

OR

2. Suppose $f : [-\pi, \pi) \to C$ is both continuous and bounded, say $|f(\theta)| \leq M$ for all θ. If $< f, e^{in\theta} >= 0$ for all $n \in Z$, prove that $f(\theta)$=0 for all $\theta \in [-\pi, \pi)$.

20.1. (i) For $k \in Z$, define the translation operator $R_k : l^2(Z) \to l^2(Z)$.

(ii) When we say a linear transformation $T : l^2(Z) \to l^2(Z)$ is translation invariant?.

(iii) Suppose $T : l^2(Z) \to l^2(Z)$ is a bounded translation invariant linear transformation. If we define $b \in l^2(Z)$ by $b = T(\delta)$, then prove that $T(z) = b * z$ for all $z \in l^2(Z)$.

OR

2. Suppose $u_l, v_l \in l^1(Z)$ for each $l \in N$ and the system matrix $A_l(\theta)$ is unitary for all $\theta \in [0, \pi)$. Define $f_1 = u_1, g_1 = u_1$ and for $l \in N, l \geq 2$, define $f_l = g_{l-1} * U^{l-1}(v_l)$ and $g_l = g_{l-1} * U^{l-1}(u_l)$. If we define V_{-l} for $l \in N$ as $V_l = [\sum\limits_{k\in Z} z(k)R_{2^l k} g_l : z = (z(k))_{k\in Z} \in l^2(Z)]$ and if $\bigcap\limits_{l\in N} V_{-l} = \{0\}$, prove that $B = \{R_{2^l k} f_l : k \in Z, l \in N\}$ is a complete orthonormal set in $l^2(Z)$.

(10 x 4 = 40)