Reg. No .... Name **19P4003**

MSc DEGREE END SEMESTER EXAMINATION - MARCH/APRIL 2019

SEMESTER 4 : MATHEMATICS

COURSE : **16P4MATT16EL : DIFFERENTIAL GEOMETRY**

(For Regular - 2017 Admission and Supplementary 2016 Admission)

Time : Three Hours Max. Marks: 75

Section A

Answer any 10 (1.5 marks each)

1. Describe the level sets of $f(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3$, for $n = 0, 1, 2$.
2. Sketch the level sets $f^{-1}(c)$ for $n = 0, 1, 2$, where $f(x_1, \ldots, x_{n+1}) = x_{n+1}$ and $c = -1, 0, 1, 2$
3. Sketch the vector field on $\mathbb{R}^2 : \mathbb{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (-2x_2, \frac{1}{2}x_1)$.
4. Find the velocity, the acceleration, and the speed of parametrized curve $\alpha(t) = (\cos 3t, \sin 3t)$.
5. Find the velocity, the acceleration, and the speed of parametrized curve $\alpha(t) = (\cos t, \sin t, 2\cos t, 2\sin t)$.
6. Show that if $\mathbb{X}$ is parallel along α, then $\mathbb{X}$ has constant length.
7. Define parametrization of a segment of the plane curve C containing p.
8. Define the circle of curvature of a plane curve.
9. Give Frenet formula for a plane curve.
10. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$, let $p \in S$, and let $\{k_1(p), \ldots, k_n(p)\}$ be the principal curvatures of S at p with corresponding orthogonal principal curvature directions $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Prove that the normal curvature $k(\mathbf{v})$ in the direction $\mathbf{v} \in S_p$ is given by $k(\mathbf{v}) = \sum_{i=1}^{n} k_i(p)(\mathbf{v} \cdot \mathbf{v}_i)^2 = \sum_{i=1}^{n} k_i(p) \cos^2 \theta_i$ where $\theta_i = \cos^{-1}(\mathbf{v} \cdot \mathbf{v}_i)$.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. Find the integral curve through $p = (x_1, x_2) = (1, 1)$ of the vector field $\mathbb{X}(p) = (p, x_2, x_1)$.
12. Let U be an open set in $\mathbb{R}^{n+1}$. $\mathbb{X}$ be a smooth vector field on U. Suppose $\alpha : I \to U$ is an integral curve of $\mathbb{X}$ with $\alpha(0) = \alpha(t_0)$ for some $t_0 \in I,\ t_0 \neq 0$. Show that α is periodic.

13. Let S be a 2-surface in $\mathbb{R}^3$ and let $\alpha : I \to S$ be a geodesic in S with $\dot{\alpha} \neq 0$. Prove that a vector field $\mathbb{X}$ tangent to S along α is parallel along α if and only if both $||\mathbb{X}||$ and the angle between $\mathbb{X}$ and α are constant along α.

14. Compute $\nabla_v f$ where $f(x_1, x_2) = 2x_1^2 + 3x_2^2$, $v = (1, 0, 2, 1)$.

15. Describe the geometric meaning of Weingarten map.

16. Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self-adjoint linear transformation on V. Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \to \mathbb{R}$ by $f(v) = L(v) \cdot v$. Suppose$f$ is staionary at $v_0 \in S$. Prove that $L(v_0) = f(v_0)v_0$.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

OR

2. Consider the vector field $\mathbb{X}(x_1, x_2) = (x_1, x_2, x_2, x_1)$ on $\mathbb{R}^2$. For $t \in \mathbb{R}$ and $p \in \mathbb{R}^2$, let $\varphi_t(p) = \alpha_p(t)$ where α_p is the maximal integral curve of $\mathbb{X}$ through p. Prove that $t \mapsto \varphi_t$ is a homomorphism from the additive group of real numbers into the group of one to one transformations of the plane.

18.1. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$, and let α be a piecewise smooth parametrized curve from p to q. Prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

OR

2. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0 \ \forall p \in S$. Prove that the Gauss map maps S onto the unit sphere S^n.

19.1. Prove that the Weingarten map Lp is self-adjoint.

OR

2. Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrization of C. Then prove that
(i) β is either one to one or periodic.
(ii) β is periodic if and only if C is compact.

20.1. (i) Find the Gaussian curvature of $\phi(t, \theta) = (\cos\theta, \sin\theta, t)$
(ii) Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite.

OR

2. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $\mathbf{v}$ be a unit vector in S_p, $p \in S$. Then prove that
(i) There exists an open set $V \subset \mathbb{R}^{n+1}$ containing p such that $S \cap \mathcal{N}(\mathbf{v}) \cap V$ is a plane curve.
(ii) The curvature at p of this curve is equal to the normal curvature $k(v)$.

(10 x 4 = 40)