19P3053

Reg. No .... Name ....

M. Sc DEGREE END SEMESTER EXAMINATION - OCTOBER 2019

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT15 : NUMBER THEORY**

(For Regular - 2018 Admission and Supplementary - 2016/2017 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer all Questions (1.5 marks each)

1. Give a multiplicative function which is not completely multiplicative. Explain.
2. Show that the divisor functions are multiplicative.
3. Solve the congruence $25x \equiv 15(\mod 120)$.
4. State and prove Wilson's theorem.
5. Let D be a domain. Prove that x is a unit if and only if $x|1$.
6. Show by an example that there exists a ring which is not Noetherian.
7. Let D be a domain and x and y non-zero elements of D. Prove that x and y are associates if and only if $\langle x\rangle = \langle y\rangle$.
8. Define fractional ideal. Prove that the fractional ideal $\mathfrak{a} \subset \mathfrak{O}$ is an ideal.
9. True or false: every ideal of $\mathfrak{O}$ has an $\mathbb{Z} - basis$. Justify?
10. Prove that $\mathbb{R}[x, y]/\langle x\rangle$ is isomorphic(as rings) to $\mathbb{R}[y]$.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. Prove that if both g and $f * g$ are multiplicative, then f is also multiplicative.
12. Prove that the set of lattice points visible from the origin has density $6/\pi^2$
13. If $x \geq 1$, prove that

    1. $\sum_{n>x} \frac{1}{n^s} = O(x^{1-s})$ if $s > 1$.
    2. $\sum_{n\leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$ if $\alpha \geq 0$.

14. Prove that for $x \geq 2$, $\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t\log^2 t}dt.$
15. Prove that the units $U(R)$ of a commutative ring R with unity form a group under multiplication.
16. Prove that every non zero prime ideal of $\mathfrak{O}$ is maximal.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. State and prove Euler's summation formula.

OR

2. Prove that $\sum_{p \le x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$ for $x \ge 2$ where the sum is extended over all primes $\le x$.

18.1. Prove that the following statements are equivalent

1. $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$
2. $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$
3. $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1.$

OR

2. If p is odd, $p > 1$, prove that

1. $1^2 3^2 5^2 \ldots (p-2)^2 = (-1)^{P+1)/2} (\mod p)$
2. $2^2 4^2 6^2 \ldots (p-1)^2 = (-1)^{(P+1)/2} (\mod p).$

19.1. Prove that in a domain, in which factorization into irreducibles is possible, factorization is unique if and only if every irreducible is prime.

OR

2. Define Euclidean quadratic Field. Prove that the ring of integers $\mathfrak{O}$ of $\mathbb{Q}(\sqrt{d})$ is Euclidean for $d = -2, -11.$

20.1. Prove that the non-zero fractional ideals of $\mathfrak{O}$ form an abelian group under multiplication.

OR

2. Express $\langle 18 \rangle$ as a product of prime ideals.

(10 x 4 = 40)