19P3018

Reg. No .... Name ....

M. Sc DEGREE END SEMESTER EXAMINATION - OCTOBER 2019

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT12 : ADVANCED FUNCTIONAL ANALYSIS**

(For Regular - 2018 Admission and Supplementary - 2016/2017 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer all Questions (1.5 mark each)

1. Let $T : X \to Y$ be a bounded linear operator, where X and Y are normed spaces. If $x_n \stackrel{w}{\longrightarrow} x$ then prove that $(T(x_n))$ is strongly operator convergent with limit Tx.
2. If $T : X \to X$ is a contraction on a metric space X, show that T^n $(n \in N)$ is a contraction on X.
3. Let $T_n : l^2 \to l^2$ be an operator such that $T(x) = (0, 0, \ldots, 0, \xi_1, \xi_2, \ldots, \xi_n \ldots)n$ zeroes, where $x = (\xi_j) \in l^2$. Prove that T_n is linear and bounded.
4. Let A be a complex Banach algebra with identity `e' and $w \in A$ be such that $||w|| < \frac{1}{2}$. Then prove that $||(e - w)^{-1} - e - w|| \leq 2||w||^2$.
5. Prove that the set G of all invertible elements of an algebra A with identity 'e' is a group.
6. If X is an infinite dimensional normed space, prove that the identity operator on X is not compact.
7. Let $T : X \to Y$ be a linear operator where X and Y are normed spaces. If X is finite dimensional then prove that T is compact.
8. Define a positive operator on a complex Hilbert space.
9. Suppose S and T are two bounded self adjoint linear operators on a complex Hilbert space H such that $S \leq T$ and $T \leq S$. Then prove that $S = T$.
10. Let $T : H \to H$ be a bounded self adjoint linear operator on a compact Hilbert space H. Prove that the eigen vectors corresponding to different eigen values are orthogonal.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. (a) If $x_n \stackrel{w}{\to} x_0$ in a normed space X, show that $x_0 \in \bar{Y}$, where $Y = \text{span}\,(x_n)$.
(b) If (x_n) is a weakly convergent sequence in a normed space X, say $x_n \stackrel{w}{\to} x_0$, show that there is a sequence (y_m) of linear combinations of elements of (x_n) which converges strongly to x_0.
12. Let $T_n \in B(X, Y)$, where X is a Banach space and Y a normed space. If (T_n) is strongly operator convergent with limit T, then prove that $T \in B(X, Y)$. If X is only a normed space (ie., X is not complete), will T be bounded always?. Justify.
13. Prove that all matrices representing a given linear operator $T : X \to X$, where X is a finite dimensional normed space, with reference to different bases for X have the same eigen values.
14. (a) Define an operator function. Is the resolvent operator $R_\lambda(T)$ of a bounded linear operator $T : X \to X$ on a complex Banach space X an operator function. Justify
(b) When an operator function S is locally holomorphic ?
(c) Prove that the resolvent $R_\lambda(T)$ is locally holomorphic on $\rho(T)$
15. Let X and Y be two normed spaces and $T : X \to Y$ a linear operator. Then prove that T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.
16. Prove that the spectrum $\sigma(T)$ of a bounded self adjoint linear operator $T : H \to H$ on a complex Hilbert space H lies is the interval $[m, M]$, where $m = \inf_{||x||=1} \langle Tx, x\rangle$ and $M = \sup_{||x||=1} \langle Tx, x\rangle$.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. (a) Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subseteq X$ and X and Y are normed spaces. Then prove that T is closed if and only if it has the following properties if $(x_n) \to x$, where $(x_n) \subseteq D(T)$, and $(Tx_n) \to y$, then $x \in D(T)$ and $Tx = y$.
(b) State and prove Banach Fixed point theorem.

OR

2. (a) Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Then prove the following.(i) If $\mathcal{D}(T)$ is a closed subset of X, then T is closed. (ii) If T is closed and Y is complete, then $\mathcal{D}(T)$ is a closed subset of X.
(b) Let T be a closed linear operator from a Banach space X into a normed space Y. If T^{-1} exists and T^{-1} is bounded, then prove that $R(T)$ is a closed subset of Y.

18.1. State and prove spectral mapping theorem for polynomials

OR

2. (a) Prove that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open.
(b) Prove that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is compact.

19.1. (a) Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent, then prove that the uniform operator limit is compact.
(b) Prove that the above result need not hold if we replace the uniform operator convergence by strong operator convergence.

OR

2. (a) Prove that the range $R(T)$ of a compact linear operator $T : X \to Y$ is separable, where X and Y are normed spaces.
(b) Prove that a compact linear operator $T : X \to Y$ from a normed space X into a Banach space Y has a compact linear extension $\tilde{T} : \hat{X} \to Y$, where $\hat{X}$ is the completion of X.

20.1. (a) Define a monotone sequence (T_n) of bounded self adjoint linear operators $T_n : H \to H$ on a complex Hilbert space H.
(b) Let (T_n) be a sequence of bounded self adjoint linear operators on a complex Hilbert space H such that $T_1 \leq T_2 \leq T_3 \leq \ldots \leq K$. Suppose that T_j commutes with K and with every T_m. Then prove that (T_n) is strongly operator convergent. $(T_n x \to Tx$ for all $x \in H)$ and the limit operator T is a bounded self adjoint linear operator satisfying $T \leq K$.

OR

2. Let (P_n) be a monotonic increasing sequence of projections P_n defined on a Hilbert space H. Then prove that

$$P(H) = \bigcup_{n=1}^{\infty} \overline{P}_n(H)$$

(a) P projects H onto

$$N(P) = \bigcap_{n=1}^{\infty} N(P_n)$$

(b) P has the null space.

(10 x 4 = 40)