Reg. No .... Name **19P2043**

MSc DEGREE END SEMESTER EXAMINATION - MARCH/APRIL 2019

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT09 : FUNCTIONAL ANALYSIS**

(For Regular - 2018 Admission and Supplementary - 2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer all the following (1.5 marks each)

1. Define a bounded linear operator. Give an example.
2. If Y is a subspace of a vector space X and f_n a linear functional on X such that $f(Y)$ is not the whole scalar field of X, show that $f(y) = 0$ for all $y \in Y$.
3. State and prove Pythagorean theorem in an inner product space.
4. In an inner product space, if $\langle x, u\rangle = \langle x, v\rangle$ for all x, prove that $u = v$
5. If Y is a closed subspace of a Hilbert space H, prove that $Y^{\perp}$ is also a closed subspace of H.
6. If Y is a closed subspace of a Hilbert space H, prove that $Y \cap Y^{\perp} = \{0\}$.
7. Prove that every bounded linear functional f on l^2 can be represented in the form $f(x) = \sum_{j=1}^{\infty} \xi_i \overline{\eta_i}, x = (\xi_j) \in l^2$ and $z = (n_j)$ is a fixed element of l^2.
8. If $U : H \to H$ is a unitary operator on a Hilbert space H, then prove that U is isometric.
9. If $T \in B(X, Y)$, where X and Y are normed spaces and α is a scalar, prove that $(\alpha T)^{\times} = \alpha T^{\times}$
10. Prove that $||.||$ is a sub linear functional.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. Prove that the inverse of a bounded linear operator, if it exists, need not be bounded.
12. Let $T : C[0, 1] \to C[0, 1]$ be defined by $y(t) = \int_0^t x(s)ds = T(x(t))$. Find $R(T)$ and $T^{-1} : R(T) \to C[0, 1]$. Is T^{-1} linear and bounded? Justify.
13. Prove that l^p with $p \neq 2$ is not an inner product space. Is l^2 an inner product space? Justify.
14. Prove that in an inner product space X, $x_n \to x$ and $y_n \to y$ imply (a) $\langle x_n, y\rangle \to \langle x, y\rangle$, (b) $\langle x, y_n\rangle \to \langle x, y\rangle$ and (c) $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

15. If h is a bounded sesquilinear functional defined as $X \times Y$ (X and Y are normed spaces), prove that

a. $x_n \to x$ implies $h(x_n, y) \to h(x, y)$
b. $y_n \to y$ implies $h(x, y_n) \to h(x, y)$
c. $x_n \to x$ and $y_n \to y$ imply $h(x_n, y_n) \to h(x, y)$, when (x_n) is a sequence in X, (y_n) is a sequence in Y, $x \in X$ and $y \in Y$

16. If X is a normed space, $x \in X$ and g_x is a functional defined on X' by $g_x(f) = f(x)$ for all $f \in X'$, then prove that g_x is bounded linear and $||g_x|| = ||x||$

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. a. Prove that on a finite dimensional vector space X any norm $||.||$ is equivalent to any other norm.
b. If a normed space X has the property that the closed unit ball $M = \{x \in X| ||x|| \leq 1\}$ is compact, than prove that X is finite dimensional.

OR

2. a. Prove that every linear operator defined as a finite dimensional named space X is bounded. Will all linear operators $T : R \to R$ be bounded? Justify.
b. Let $T : D(T) \to Y$ be a bounded linear operator, where $D(T) \subset X$, X is a normed space and Y is a Banach space. Then prove that T has an extension $\tilde{T} : \overline{D(T)} \to Y$, where $\tilde{T}$ is bounded linear and $||\tilde{T}|| = ||T||$.

18.1. a. If X is an inner product space, prove that

$$|\langle x, y\rangle| \leq ||x||\ ||y|| \text{ for all } x, y \in X$$

Also prove that the equality holds if and only if $\{x, y\}$ is a linearly dependent set.
b. Prove that the norm induced by the inner product in an inner product space X satisfies $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$, where the equality holds if and only if either $y = 0$ or $x = cy$ (c is real and non-negative).

OR

2. a. Let Y be a subspace of a Hilbert space H. Then prove that:
(i) If Y is finite dimensional, then Y is complete.
(ii) If H is separable, so is Y.
b. Show that for a sequence (x_n) is an inner product space X the conditions

$$||x_n|| \to ||x|| \text{ and } \langle x_n, x\rangle \to \langle x, x\rangle$$

imply the convergence $x_n \to x$.

19.1. a. Let X be an inter product space and $M \neq \phi$ be a convex subset which is complete. Then for any $x \in X$ prove that there exists a unique $y \in M$ such that $\delta = \inf_{\tilde{y} \in M} ||x - \tilde{y}|| = ||x - y||$

b. If M is a complete subspace Y, then prove that $z = x - y$ is orthogonal to Y

OR

2. a. If H is a separable Hilbert space, then prove that every orthonormal set in H is countable

b. If a Hilbert space H contains a total orthonormal sequence, then prove that H is separable.

c. Let H be a Hilbert space, $S : H \to H$ and $T : H \to H$ two bounded linear operators. Then prove that

i. $(ST)^* = T^* S^*$

ii. $(\alpha T)^* = \overline{\alpha}\, T^*$ (α is a scalar)

20.1. a. Define a sub linear functional on a real vector space X

b. State and prove Hahn Banach theorem for a real vector space.

OR

2. a. State and prove the generalized Hahn Banach theorem.

b. If p is a real valued functional defined on a vector space X, satisfying

$$p(x + y) \leq p(x) + p(y)$$

and $p(\alpha x) = |\alpha| p(x)$ for all $x, y \in X$ and for all scalars α, then prove that $p(0) = 0$ and $p(x) \geq 0$ for all $x \in X$.

(10 x 4 = 40)