Reg. No .... Name **19P2003**

MSc DEGREE END SEMESTER EXAMINATION - MARCH/APRIL 2019

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT06 : ABSTRACT ALGEBRA**

(For Regular - 2018 Admission and Supplementary - 2017/2016 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer the following 10 (1.5 marks each)

1. Find the maximum possible order of some element in $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$.
2. What are the possible numbers of Sylow 3-subgroups of a group of order 255?
3. How many abelian groups up-to isomorphism are there of order 15? How many non abelian groups upto isomorphism are there of order 15?Justify your answer.
4. Does every abelian group of order divisible by 6 contain a cyclic subgroup of order 6? Justify your answer.
5. Suppose that R is a ring and $f(x)$ and $g(x)$ in $R[x]$ are of degrees 3 and 4, respectively.Is $f(x)g(x)$ always of degree 7? Justify your answer.
6. Find deg$(\sqrt{2}, \mathbb{R})$.Is it equal to deg$(\sqrt{2}, \mathbb{Q})$? Justify your answer.
7. How many fields are there (upto isomorphism) of order 6? Justify your answer.
8. State the Isomorphism Extension Theorem.
9. Define normal extension of a field F? Give an example.
10. What are the elements of $G(E/F)$, the group of E over F? When is $G(E/F)$ known as the Galois group of E over F?

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. Show that a group of order 48 is not simple.
12. (a) Find the order of the torsion subgroup of $\mathbb{Z}_4 \times \mathbb{Z} \times \mathbb{Z}_3$
    (b) Find the torsion subgroup of the multiplicative group $\mathbb{R}^*$ of nonzero real numbers.
13. Show that no element of $\mathbb{Q}(\sqrt{2})$ is a zero of $x^3 - 2$.Find an extension of $\mathbb{Q}(\sqrt{2})$ which contains a zero of this polynomial.
14. Let E be an extension field of F and let $\alpha, \beta \in E$.Suppose α is transcendental over F but algebraic over $F(\beta)$.Show that β is algebraic over $F(\alpha)$.
15. Show that the set of all constructible real numbers forms a subfield of $\mathbb{R}$.
16. If K is a finite extension of E and E is a finite extension of F, show that K is separable over F if and only if K is separable over E and E is separable over F.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. (a). Let G be a group containing normal subgroups H and K such that $H \cap K = \{e\}$ and $H \vee K = G$. Show that G is isomorphic to $H \times K$.
(b). Define the class equation of a group G. Using it show that the center of a finite non-trivial p-group G is non-trivial.
(c). Show that a group of order 81 is solvable.

OR

2. (a). Show that every group of order 1645 is cyclic.
(b). Show that every group of order 30 contains a subgroup of order 15.

18.1. (a). Define principal ideal.Show that if F is a field, then every ideal in $F[x]$ is principal.
(b). Show that an ideal $\langle p(x) \rangle \neq \{0\}$ is maximal if and only if $p(x)$ is irreducible over F.
(c). Let $p(x)$ be an irreducible polynomial in $F[x]$. If $p(x)$ divides $r(x)s(x)$ for $r(x), s(x) \in F[x]$, show that either $p(x)$ divides $r(x)$ or $p(x)$ divides $s(x)$.

OR

2. (a). Show that a finite extension field E of a field F is an algebraic extension of F.
(b). If E is a finite extension of a field F, and K is a finite extension of E, show that K is a finite extension of F, and $[K : F] = [K : E][E : F]$.
(c). If E is a finite extension of a field F, $\alpha \in E$ is algebraic over F, and $\beta \in F(\alpha)$, show that $\deg(\beta, F)$ divides $\deg(\alpha, F)$.

19.1. (a). Show that trisecting the angle is impossible.
(b). Show that the field F of constructible real numbers consists precisely of all real numbers that can be obtained from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations.

OR

2. (a). Show that if F is a field of prime characteristic p with algebraic closure $\overline{F}$, then $x^{p^n} - x$ has p^n distinct zeroes in $\overline{F}$.
(b). Show that if F is a field of prime characteristic p, then $(\alpha + \beta)^{p^n} = \alpha^{p^n} + \beta^{p^n}$ for all $\alpha, \beta \in F$ and all positive integers n.

20.1. (a). If E is a finite extension of F, then show that $\{E : F\}$ divides $[E : F]$.
(b). Show that $\alpha \in \overline{F}$ is separable over F if and only if $irr(\alpha, F)$ has all zeroes of multiplicity 1.
(c). If K is a finite extension of E and E is a finite extension of F, i.e. $F \leq E \leq K$, show that K is separable over F if and only if K is separable over E and E is separable over F.

OR

2. Find the splitting field K of $x^4 - 2$ over $\mathbb{Q}$. Compute $G(K/\mathbb{Q})$, find its subgroups and the corresponding fixed fields and draw the subgroup and subfield lattice diagrams.

(10 x 4 = 40)