19P1031

Reg. No .... Name ....

M. Sc DEGREE END SEMESTER EXAMINATION - OCTOBER 2019

SEMESTER 1 : MATHEMATICS

COURSE : **16P1MATT03 : MEASURE THEORY AND INTEGRATION**

(For Regular - 2019 Admission and Supplementary - 2016/2017/2018 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer all Questions (1.5 mark each)

1. Prove that the outer measure of the set of all rationals in [0,1] is zero.
2. If $m^*A = 0$, then prove that $m^*(A \cup B) = m^*B$ for any set B.
3. Give an example of a decreasing sequence $< E_n >$ of measurable sets such that $m(\cap_1^\infty E_n) \neq \lim mE_n$.
4. Define a step function. Give an example.
5. If f is integrable over a measurable set E of finite measure and $A \leq f \leq B$, then prove that $AmE \leq \int_E f \leq BmE$. Hence, prove that there exists $A \leq k \leq B$ such that $\int_E f = kmE$. Deduce that $\int_a^b f = k(b-a)$.
6. If f is integrable, then prove that f is finite valued a.e.
7. Let `c' be a constant and f be a measurable function defined on X, where $(X, \mathcal{B})$ is a measurable space.
   Then prove that cf and $f + c$ are measurable.
8. If μ is a measure on an algebra $\mathfrak{A}$ and μ^* is the outer measure defined by μ, prove that $\mu^*A = \mu A$ if $A \in \mathfrak{A}$.
9. Prove that every finite measure is a σ-finite measure but the converse of it is not true.
10. Let μ and ν be complete measures. Show that $\mu \times \nu$ need not be complete.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. (a) Define the binary operation sum modulo 1 ($\overset{\circ}{+}$) on [0,1).
    (b) Prove that $\overset{\circ}{+}$ is associative and commutative.
    (c) What is the inverse of any $x \in [0,1)$ under $\overset{\circ}{+}$?.
12. (a) Define cantor ternary set. Is it measurable? Justify. (b) Show that Cantor ternary set has measure zero.
13. Let $\langle u_n \rangle$ be a sequence of non-negative measurable functions and let $f = \sum_1^\infty u_n$. Then prove that $\int f = \sum_1^\infty \int u_n$.
14. Let f and g be integrable over E. Then prove that
    (a) The function cf is integrable over E and $\int_E cf = c\int_E f$ (c is a constant)
    (b) The function $f + g$ is integrable over E and
    $$\int_E (f+g) = \int_E f + \int_E g.$$

15. Let $\mathfrak{A}$ be an algebra of subsets of a space X. If $A \in \mathfrak{A}$ and if $\langle A_i \rangle$ is any sequence of sets in $\mathfrak{A}$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$, then prove that $\mu A \leq \sum_{i=1}^{\infty} \mu A_i$.

16. Prove that $\mathcal{S} \times \mathcal{J} = \mathcal{M}_\circ(\mathcal{E})$.

(5 x 4 = 20)

Section C
Answer any 4 (10 marks each)

17.1. (a) Let f be an extended real valued function whose domain is a measurable set. Prove that the following statements are equivalent
(i) for each real number α, $\{x : f(x) > \alpha\}$ is measurable.
(ii) for each real number α, $\{x : f(x) \geq \alpha\}$ is measurable.
(iii) for each real number α, $\{x : f(x) < \alpha\}$ is measurable.
(iv) for each real number α, $\{x : f(x) \leq \alpha\}$ is measurable.
(b) If f is Lebesgue measurable, prove that $\{x : f(x) = \alpha\}$ is measurable for all extended real numbers α.

OR

2. (a) If f is a measurable function, then prove that $\mathcal{M} = \{E : f^{-1}(E) \text{ is measurable}\}$ is a σ-algebra.
(b) If B is a Borel set, prove that $f^{-1}(B)$ is measurable.
(c) If $\langle f_n \rangle$ is a sequence of measurable functions (with the same domain), then prove that
(i) $\sup\{f_1, f_2, \ldots, f_n\}$ is measurable.
(ii) $\sup_n f_n$ is measurable.
(iii) $\overline{\lim} f_n$ is measurable.

18.1. (a) Define Riemann integral of a bounded function over a finite closed integral $[a, b]$ interms of step functions.
(b) Define Lebesgue integral of a bounded measurable function defined on a measurable set E with mE finite.
(c) Let f be a bounded function defined an $[a, b]$. If f is Riemann integrable, then prove that it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)dx.$$

OR

2. (a) State and prove Bounded Convergence theorem. (b) State and prove Fatou's lemma.

19.1. (a) State and prove Hahn decomposition theorem.
(b) Give an example to show that the Hahn decomposition need not be unique.

OR

2. (a) Let (X, B, μ) be a measuer space and f be a measurable function defined on X such that $\int f d\mu$ is defined. Prove that the set function ν defined on B by $\nu E = \int_E f d\mu$ is a signed measure.
(b) Find a Hahn decomposition of X w.r.t. ν
(c) Find a Jordan decomposition of ν.

20.1. If $E \in \mathcal{S} \times \mathcal{J}$, then prove that for each $x \in X$ and $y \in Y$, $E_x \in \mathcal{J}$ and $E^y \in \mathcal{S}$.

OR

2. Let f be a non-negative $\mathcal{S} \times \mathcal{J}$ measurable function and let $\phi(x) = \int_Y f_x d\nu$, $\psi(y) = \int_X f^y d\mu$ for each $x \in X$ and $y \in Y$. Then prove that ϕ is $\mathcal{S}$-measurable and ψ is $\mathcal{J}$-measurable and $\int_X \phi d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y \psi d\nu$.

(10 x 4 = 40)