19P1003

Reg. No .... Name ....

M. Sc DEGREE END SEMESTER EXAMINATION - OCTOBER 2019

SEMESTER 1 : MATHEMATICS

COURSE : **16P1MATT01 : LINEAR ALGEBRA**

(For Regular - 2019 Admission and Supplementary - 2016/2017/2018 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer all Questions (1.5 mark each)

1. Is the set of vectors $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $a_1 \geq 0$ a subspace of $\mathbb{R}^n$?
2. Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Show that the set obtained by adjoining β to S is linearly independent.
3. Prove that the space of all $m \times n$ matrices over the field F has dimension mn, by exhibiting a basis for this space.
4. Define a non-singular transformation. Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, y)$ is non-singular.
5. Define range,rank, null space, and nullity of a linear transformation.
6. Define hyperspace in a vector space. Give an example.
7. Let D be a 2-linear function with the property that $D(A) = 0$ for all 2×2 matrices A over K having equal rows. Show that D is alternating.
8. Show that similar matrices have the same characteristic polynomial.
9. Define the terms characteristic value, characteristic vector and characteristic space with respect to a linear operator T on a vector space V.
10. Define invariant subspace with an example. Also state a necessary condition for a subspace to be invariant.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions $f(-x) = -f(x)$.
(a) Prove that V_e and V_o are subspaces of V.
(b) Prove that $V_e + V_o = V$
(c) Prove that $V_e \cap V_o = \{0\}$.
12. Let A be an $n \times n$ matrix over a field F and suppose that the row vectors of A form a linearly independent set of vectors in F^n. Show that A is invertible.
13. Let $\mathscr{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 = (2, 2, 0)$. Find the dual basis of $\mathscr{B}$.
14. Show that $\{(1, 2), (3, 4)\}$ is a basis for $\mathbb{R}^2$. Let T be the unique linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $T(1, 2) = (3, 2, 1)$ and $T(3, 4) = (6, 5, 4)$. Find $T(1, 0)$
15. Let A be an $n \times n$ matrix with λ as an eigen value. Show that,
(a) $k + \lambda$ is an eigen value of $A + kI$.
(b) If A is non-singular, $\frac{1}{\lambda}$ is an eigen value of A^{-1}.
16. Let V be a finite dimensional vector space over a field F and T be a linear operator on V. Prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

(5 x 4 = 20)

Section C
Answer any 4 (10 marks each)

17.1. (a)Let V be the set of all complex valued functions on the real line such that $f(-t) = \overline{f(t)}$, where $\overline{f(t)}$ is the conjugate of $f(t)$. Show that V is a vector space over $\mathbb{R}$. Is V finite dimensional? Justify your answer.

(b) Let V be the space of all 2×2 matrices over $\mathbb{R}$.Let W_1 be the set of all matrices of the form $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and W_2 be the set of all matrices of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $x, y, z \in \mathbb{R}$.Show that W_1 and W_2 are subspaces.Find the intersection $W_1 \cap W_2$

OR

2. Let V be the vector space of all 2×2 matrices over the field F.Let W_1 be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.Prove that W_1 and W_2 are subspaces of V. Also find the dimensions of W_1,W_2,$W_1 + W_2$ and $W_1 \cap W_2$.

18.1. Let V be a finite- dimensional vector space over the field F. Prove that V and V^{**} are isomorphic.Further show that each basis for V^* is the dual of some basis for V.

OR

2. (a) Does there exist a linear transformation $T : R^3 \to R^2$ such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$?. Justify.

(b) Let V and W be finite-dimensional vector spaces over the field F. Prove that V and W are isomorphic if and only if $\dim V = \dim W$.

(c) Let T be the linear operator on R^2 defined by $T(x_1, x_2) = (x_1, 0)$. Compute the matrix of T relative to the ordered basis $\{(1, 1), (2, 1)\}$.

19.1. (a) Let D be an n-linear function on the space of $n \times n$ matrices over a field K. Suppose D has the property that $D(A) = 0$ whenever two adjacent rows of A are equal. Show that D is alternating.

(b) Let $n > 1$ and let D be an alternating $(n - 1)$ linear function on an $(n - 1) \times (n - 1)$ matrix over K. Show that for each $j, j = 1, \ldots, n$, the function E_j defined by $E_j(A) = \sum_{i=1}^{n} (-1)^{(i+j)} A_{ij} D_{ij}(A)$ is an alternating n-linear function on the space of $n \times n$ matrices A. If D is the determinant function, so is E_j.

OR

2. (a) If A is an $n \times n$ skew symmetric matrix with complex entries and n is odd, prove that $\det A = 0$.

(b) If A is an $n \times n$ invertible matrix over a field F, show that $\det A \neq 0$.

20.1. (a) Let T be a linear operator on a finite dimensional space V. Let $c_1, c_2, \cdots, c_k$ be the distinct characteristic values and $W_1, W_2, \cdots, W_k$ be the corresponding characteristic spaces. Prove that $\dim(W_1 + W_2 + \cdots + W_k) = \dim W_1 + \dim W_2 + \cdots + \dim W_k$.

(b) If W_1 and W_2 are subspaces of V then prove that they are independent if and only if $W_1 \cap W_2 = 0$.

OR

2. (a) If W is an invariant subspace for T , show that W is invariant under every polynomial in T. Hence show that for each $\alpha \in V$,the T-conductor $(S\alpha, W)$ is an ideal in the polynomial ring $F[X]$.

(b) Let W be an invariant subspace for T.Show that the characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T and the minimal polynomial for the restriction operator T_W divides the minimal polynomial for T.

(10 x 4 = 40)