MSC DEGREE END SEMESTER EXAMINATION MARCH 2016
SEMESTER -2: M SC MATHEMATICS
COURSE: P2MATT10- REAL ANALYSIS

Time: Three Hours **Max Marks: 75**

Part A
Answer Any Five Questions
Each Question carries 2 Marks

1. Show that the set of all discontinuities of a monotone function f on an interval $[a, b]$ is countable.
2. Show that if f is continuous on $[a, b]$ and the derivative f' is bounded on (a, b), then f is of bounded variation on $[a, b]$.
3. If $f_1, f_2 \in \Re(\alpha)$ on $[a, b]$, then show that $c_1 f_1 + c_2 f_2 \in \Re(\alpha)$ for every constants c_1, c_2 and also show that $\int_a^b (c_1 f_1 + c_2 f_2) d\alpha = c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha$.
4. If $f \in \Re(\alpha)$ on $[a, b]$, then show that $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha$.
5. Define uniform convergence and give an example.
6. State Weierstrass Approximation Theorem.
7. Give an example for a double sequence $\{a_{i,j}\}$ with $\sum_i \sum_j a_{i,j} \neq \sum_j \sum_i a_{i,j}$.
8. Define orthonormal system of functions on an interval $[a, b]$ and give an example.

Part B
Answer Any Five Questions
Each Question carries 5 Marks

9. Show that there exists a continuous function which is not of bounded variation.
10. Show that a polynomial f is of bounded variation on every compact interval $[a, b]$.
11. Show that $f \in \Re(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition $\mathbf{P}$ such that
$$U(\mathbf{P}, f, \alpha) - L(\mathbf{P}, f, \alpha) < \epsilon.$$
12. Show that if f is continuous on $[a, b]$, then $f \in \Re(\alpha)$ on $[a, b]$.
13. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
14. Given a double sequence $\{a_{i,j}\}, i, j = 1, 2, 3, \ldots$, suppose that
$$\sum_{j=1}^{\infty} |a_{i,j}| = b_i \ \ (i = 1, 2, 3, \ldots)$$
and $\sum_{i=1}^{\infty} b_i$ converges. Then show that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$.

15. Define *analytic functions* and show by an example that a real-valued function which is differentiable infinitely many times need not be analytic.

16. Introduce Dirichlet kernel $D_N(x)$ and show that

$$s_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt,$$

where $s_N(f;x)$ is the N^{th} partial sum of the Fourier series of f.

Part C

Answer Either Part I or Part II of Each Question

Each Question Carries 10 *Marks*

17. (I)Let f be of bounded variation on $[a,b]$, and assume that $c \in (a,b)$. Then show that f is of bounded variation on $[a,c]$ and $[c,b]$. Also show that

$$V_f\,(a,b) = V_f\,(a,c) + V_f\,(c,b)$$

(II)Let f be of bounded variation on $[a,b]$, and assume that $x \in (a,b]$, let $V(x) = V_f\,(a,x)$ and $V(a) = 0$. Then show that f is continuous at a point $x_0 \in [a,b]$ if and only if V is continuous at x_0.

18. (I) Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. Also, show that this need not hold if we drop the continuity or non-negativity assumption of f.
(II) Show that every bounded real-valued function on the interval $[0,1]$, which is continuous on all points outside the Cantor set, is Riemann integrable on $[0,1]$.

19. (I)Show that there exists a real continuous function on the real line which is nowhere differentiable.
(II)Show that the uniform limit of a sequence of continuous functions on a metric space E is continuous on E.

20. (I) Let $E(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}, z \in \mathbb{C}$. Show that the series converges for every complex number z. Also show that

    - $E(z+w) = E(z)E(w)$, for every $z, w \in \mathbb{C}$.
    - The derivative of $E(x)$ is $E(x)$ itself for all real x.
    - $lim_{x \to +\infty} x^n E(-x) = 0$, for every n.

    (II)Show that the complex field is algebraically complete.