Reg. No .... Name 18P3653

MSc DEGREE END SEMESTER EXAMINATION - OCTOBER 2018

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT15 : NUMBER THEORY**

(For Regular - 2017 Admission & Supplementary - 2016 Admission)

Time : Three Hours Max. Marks: 75

Section A

Answer the following (1.5 marks each)

1. Prove that the Mangoldt function is not multiplicative.
2. Prove or disprove: Dirichlet product of two completely multiplicative functions is completely multiplicative.
3. Prove that $\psi(x) = \sum_{m\leq \log_2 x} \sum_{p\leq x^{1/m}} \log p$
4. Prove that the m residue classes $\hat{1}, \hat{2}, \ldots, \hat{m}$ are disjoint and their union is the set of all integers.
5. Let D be a domain and x and y non-zero elements of D. Prove that x is irreducible if and only if $\langle x \rangle$ is maximal among the proper principal ideals of D.
6. True or false: Every Noetherian domain is unique factorization domain.? Justify.
7. Find the group of units of the ring of integers in $\mathbb{Q}\left(\sqrt{-1}\right)$.
8. Define prime ideal. Is $\langle 0 \rangle$ in a CRU a prime ideal? Justify.
9. True or false : If a non zero ideal $\mathfrak{a}$ is prime, then $\mathrm{N}(\mathfrak{a})$ is a prime. Justify.
10. Prove that $\mathbb{R}[x, y]/\langle x \rangle$ is isomorphic(as rings) to $\mathbb{R}[y]$.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. Derive formula for the divisor sum of Euler totient function.
12. Prove that if both g and $f * g$ are multiplicative, then f is also multiplicative.
13. Find all integers n such that $\varphi(n) = 12$
14. State and prove Wolstenholme's theorem.
15. Prove that the units $U(R)$ of a commutative ring R with unity form a group under multiplication.
16. Find all the ideals in $\mathbb{Z}[\sqrt{-5}]$ which contain the element 6.

(5 x 4 = 20)

Section C

Answer the following (10 marks each)

17.1. State and prove Euler's summation formula.

OR

2. Prove that the average order of $\varphi(n)$ is $3n/\pi^2$.

18.1. 1. State Chinese Reminder theorem.

2. Solve the system of congruences $x \equiv 1 (\mod 3), x \equiv 2 (\mod 4), x \equiv 3 (\mod 5)$.

OR

2. 1. Prove the converse of Wilson's theorem.

2. Find all positive integers n for which $(n-1)! + 1$ is a power of n.

19.1. Prove that a prime in a domain D is always irreducible. Is converse true? Justify.

OR

2. Define Euclidean quadratic Field. Prove that the ring of integers $\mathfrak{O}$ of $\mathbb{Q}(\sqrt{d})$ is Euclidean for $d = -2, -11$.

20.1. Prove that any non-zero ideal $\mathfrak{a}$ of $\mathfrak{O}$ can be generated by at most two elements.

OR

2. Prove that every non-zero ideal of $\mathfrak{O}$ can be written as a product of prime ideals, uniquely up to order of the factors.

(10 x 4 = 40)