Reg. No .... Name **18P3618**

MSc DEGREE END SEMESTER EXAMINATION - OCTOBER 2018

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT12 : ADVANCED FUNCTIONAL ANALYSIS**

(For Regular - 2017 Admission & Supplementary - 2016 Admission)

Time : Three Hours Max. Marks: 75

Section A
Answer any 10 (1.5 marks each)

1. Let $X = \{x \in R \mid 1 \leq x < \infty\}$, taken with the usual metric of the real line and $T : X \to X$ be defined by $Tx = x + \frac{1}{x}$. Show that $|Tx - Ty| < |x - y|$, if $x \neq y$, but T has no fixed point.
2. In a Hilbert space H, prove that $x_n \xrightarrow{w} x$ if and only if
$$\langle x_n, z\rangle \to \langle x, z\rangle \text{ for all } z \in H.$$
3. Let $T_n : l^2 \to l^2$ be an operator such that $T(x) = (0, 0, \ldots, 0, \xi_1, \xi_2, \ldots, \xi_n \ldots) n$zeroes, where $x = (\xi_j) \in l^2$. Prove that T_n is linear and bounded.
4. Define the eigen space corresponding to an eigen value λ. Also prove that the eigen space is a vector space.
5. Prove that the elements of a point spectrum of a linear operator are the eigen values of the operator.
6. If X is an infinite dimensional normed space, prove that the identity operator on X is not compact.
7. Prove that a bounded set need not be totally bounded.
8. Let T_1 and T_2 be bounded self adjoint linear operators on a complex Hilbert space H such that $T_1T_2 = T_2T_1$ and $T_2 \geq 0$. Then show that $T_1^2T_2 \geq 0$.
9. Let $Q = S^{-1}PS : H \to H,$ where S and P are bounded linear operators on H. If P is a projection and S is unitary, show that Q is a projection.
10. Define the Hilbert adjoint operator T^* of a linear operator T and prove that it is linear.

(1.5 x 10 = 15)

Section B
Answer any 4 (5 marks each)

11. (a) We know strong operator convergence need not imply uniform operator convergence. Illustrate this by considering $T_n = f_n : l^1 \to R$, where $f_n(x) = \xi_n$ and $x = (\xi_n) \in l^1$.
(b) Let $T_n \in B(X, Y)$, where X is a Banach space and Y is a normed space. If (T_n) is strongly operator convergent, using uniform boundedness theorem prove that $(||T_n||)$ is bounded.
12. Prove that in a finite dimensional normed space weak convergence implies strong convergence. Also prove that a contraction T on a metric space is a continuous mapping.
13. Define the point spectrum, the continuous spectrum and the residual spectrum of a linear

operator $T : \mathfrak{D}(T) \to X$, where X is a non-zero complex normed space and $\mathfrak{D}(T) \subset X$. If X is finite dimensional, prove that $\sigma_c(T) = \sigma_r(T) = \phi$.

14. Let A be a Banach algebra without identity. If we define $\tilde{A} = \{(x, \alpha) | x \in A, \alpha \text{ is a scalar}\}$, then prove that $\tilde{A}$ is a Banach algebra with identity under the following operations

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta),$$
$$\beta(x, \alpha) = (\beta x, \beta\alpha), (x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$
$$||(x, \alpha)|| = ||x|| + |\alpha|$$

15. Let X and Y be two normed spaces and $T : X \to Y$ a linear operator. Then prove that T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.

16. (a) Let M be the set of all bounded self adjoint linear operators on a complex Hilbert space H. Prove that the relation `$\leq$' defined on M by $T_1 \leq T_2$ if and only if $\langle T_1 x, x \rangle \leq \langle T_2 x, x \rangle \ \forall x \in H$, is a partially ordered relation.
(b) Let S and T be bounded self adjoint linear operators on a Hilbert space H. If $S \geq 0$, show that $TST \geq 0$.

(5 x 4 = 20)

Section C
Answer any 4 (10 marks each)

17.1. (a) Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subseteq X$ and X and Y are normed spaces. Then prove that T is closed if and only if it has the following properties if $(x_n) \to x$, where $(x_n) \subseteq D(T)$, and $(Tx_n) \to y$, then $x \in D(T)$ and $Tx = y$.
(b) State and prove Banach Fixed point theorem.
OR

2. (a) Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Then prove the following.(i) If $\mathcal{D}(T)$ is a closed subset of X, then T is closed.
(ii) If T is closed and Y is complete, then $\mathcal{D}(T)$ is a closed subset of X.
(b) Let T be a closed linear operator from a Banach space X into a normed space Y. If T^{-1} exists and T^{-1} is bounded, then prove that $R(T)$ is a closed subset of Y.

18.1. (a) If X is a non-zero complex Banach space and $T \in B(X, X)$, then prove that $\sigma(T) \neq \phi$.
(b) If $T \in B(X, X)$, where X is a non-zero complex Banach space, then prove that

$$r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{||T^n||}$$

OR

2. (a) Let $T : X \to X$ be a bounded linear operator on a complex Banach space X.
Prove that for any $\lambda_0 \in \rho(T)$, $R_\lambda(T)$ has the representation
$R_\lambda(T) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}(T)^{j+1}$ and the series is absolutely convergent within the open disc given by $|\lambda - \lambda_0| < \frac{1}{||R_{\lambda_0}(T)||}$
(b) Let $T \in B(X, X)$ where X is a complex Banach space. If $\lambda, \mu \in \rho(T)$, prove the following
(i) $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda$

(ii) R_λ commutes with any $S \in B(X, X)$ satisfying $ST = TS$
(iii) $R_\lambda R_\mu = R_\mu R_\lambda$.

19.1. (a) Let X and Y be normed spaces and $T : X \to Y$ a compact linear operator. Suppose (x_n) in X is weakly convergent, say $x_n \stackrel{w}{\to} x$. Then prove that (Tx_n) converges strongly to Tx.
(b) Show that $T : l^\infty \to l^\infty$ defined by $Tx = y$, where $x = (\xi_j) \in l^\infty$ and $y = (n_j)$, $n_j = \frac{\xi_j}{j}$, is compact linear.

OR

2. (a) Let Y be a Banach space and $T_n : X \to Y$, $n = 12, 3, \ldots$ be operators of finite rank. If (T_n) is uniformly operator convergent to T, show that T is compact.
(b) Show that the projection of a Hilbert space H onto a finite dimensional subspace of H is compact.
(c) Show that $T : l^2 \to l^2$ defined by $Tx = y = (n_j)$, $n_j = \frac{\xi_j}{2^j}$, $x = (\xi_j)$, is compact.

20.1. (a) For any projection P on a Hilbert space H, prove the following
(i) $\langle Px, x\rangle = ||Px||^2$
(ii) $P \geq 0$
(iii) $||P|| \leq 1$; $||P|| = 1$ if $P(H) \neq \{0\}$.
(b) (i) If P_1 and P_2 are two projections on a Hilbert space H, then prove that $P = P_1P_2$ is a projection on H if and only if $P_1P_2 = P_2P_1$. In such a case prove that P projects H onto $Y = Y_1 \cap Y_2$, where $Y_j = P_j(H)$; $j = 1, 2$ (ii) Prove that two closed subspaces Y and V of H are orthogonal if and only if their corresponding projections satisfy $P_Y P_V = 0$.

OR

2. Let (P_n) be a monotone increasing sequence of projections P_n on a Hilbert space H.
(a) Show that (P_n) is strongly operator convergent $(P_n x \to P_x \forall x \in H)$ and the limit operator P is a projection on H
(b) Prove that $P(H) = \overline{\bigcup_{n=1}^{\infty} P_n(H)}$
(c) Prove that $N(P) = \bigcap_{n=1}^{\infty} N(P_n)$

(10 x 4 = 40)