Reg. No .... Name **18P403**

MSc DEGREE END SEMESTER EXAMINATION - MARCH 2018

SEMESTER 4 : MATHEMATICS

COURSE : **16P4MATT16EL ; DIFFERENTIAL GEOMETRY**

(For Regular - 2016 admission)

Time : Three Hours Max. Marks: 75

Section A
Answer any 10 (1.5 marks each)

1. Draw the level curve of $f(x,y)=x^2-y^2$ at $c=1$.
2. Describe the graphs and level sets(level curves) of $f(x_1,x_2)=x_1$.
3. Sketch the vector field on $\mathbb{R}^2$: $\mathbb{X}(p)=(p,X(p))$ where $X(x_1,x_2)=(-2x_2,\frac{1}{2}x_1)$.
4. Prove that the geodesics have constant speed.
5. Find the velocity, the acceleration, and the speed of parametrized curve $\alpha(t)=(t,t^2)$
6. Compute $\nabla_{\mathbf{v}}\mathbb{X}$ where $\mathbb{X}(x_1,x_2)=(x_1,x_2,x_1x_2,x_2^2)$ and $\mathbf{v}=(1,0,0,1)\in\mathbb{R}^{n+1}_p$.
7. Define Shape operator .
8. Define the circle of curvature of a plane curve.
9. Find the length of the parameterized curve $\alpha(t)=(\cos 3t,\sin 3t,4t),\ I=[-1,1]$
10. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$, let $p\in S$, and let $\{k_1(p),\ldots,k_n(p)\}$ be the principal curvatures of S at p with corresponding orthogonal principal curvature directions $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$. Prove that the normal curvature $k(\mathbf{v})$ in the direction $\mathbf{v}\in S_p$ is given by $k(\mathbf{v})=\sum_{i=1}^{n}k_i(p)(\mathbf{v}\cdot\mathbf{v}_i)^2=\sum_{i=1}^{n}k_i(p)\cos^2\theta_i$ where $\theta_i=\cos^{-1}(\mathbf{v}\cdot\mathbf{v}_i)$.

(1.5 x 10 = 15)

Section B
Answer any 4 (5 marks each)

11. Determine whether the vector field $\mathbb{X}(x_1,x_2)=(x_1,x_2,1,0)$ where $U=\mathbb{R}^2$ is complete or not.
12. State and prove the existence of Lagrange multiplier.
13. Prove that the local parameterization is unique upto a reparameterization.
14. Find the length of the parameterized curve $\alpha(t)=(\cos t,\sin t,\cos t,\sin t),\ I=[0,2\pi]$
15. State and prove any three properties of the differentiation of a vector fields with respect to a vector.
16. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$. Prove that the Gauss-Kronecker curvature $K(p)$ of S at p is non-zero for all $\in S$ if and only if the second fundamental form $\mathscr{S}_p$ of S at p is definite for all $p\in S$.

(5 x 4 = 20)

Section C
Answer any 4 (10 marks each)

17.1. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f:U\to\mathbb{R}$ be smooth. Let $p\in U$ be a regular point of f, and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

OR

2. Consider the vector field $\mathbb{X}(x_1, x_2) = (x_1, x_2, x_2, x_1)$ on $\mathbb{R}^2$. For $t \in \mathbb{R}$ and $p \in \mathbb{R}^2$, let $\varphi_t(p) = \alpha_p(t)$ where α_p is the maximal integral curve of $\mathbb{X}$ through p. Prove that $t \mapsto \varphi_t$ is a homomorphism from the additive group of real numbers into the group of one to one transformations of the plane.

18.1. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$, and let α be a piecewise smooth parametrized curve from p to q. Prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

OR

2. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$, and let $v \in S_{\alpha(t_0)}$. Show that there exists a unique vector field $\mathbb{V}$, tangent to S along α which is parallel and has $\mathbb{V}(t_0) = v$.

19.1. Prove that the Weingarten map Lp is self-adjoint.

OR

2. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by $\eta = -\frac{x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2$. Prove that for $\alpha : [a, b] \to \mathbb{R}^2 - \{0\}$ any closed piecewise smooth parameterized curve in $\mathbb{R}^2 - \{0\}$, $\int_\alpha \eta = 2\pi k$ for some integer k.

20.1. Find the Gaussian curvature for

1. $\psi(\theta, \phi) = ((a + b\cos\phi)\cos\theta), (a + b\cos\phi)\sin\theta), \sin\phi$.
2. $x_1^2 + x_2^2 - x_3^2 = 0,\ x_3 > 0$

OR

2. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $\mathbf{v}$ be a unit vector in $S_p,\ p \in S$. Then prove that

1. there exists an open set $V \subset \mathbb{R}^{n+1}$ containing p such that $S \cap \mathcal{N}(\mathbf{v}) \cap V$ is a plane curve.
2. the curvature at p of this curve is equal to the normal curvature $k(v)$.

(10 x 4 = 40)