Reg. No .................... Name .............................. **18P243**

**MSc DEGREE END SEMESTER EXAMINATION- APRIL 2018**

**SEMESTER 2 : MATHEMATICS**

COURSE : **16P2MATT09 ; FUNCTIONAL ANALYSIS**

***(For Regular - 2017 Admission & Supplementary - 2016 Admission)***

Time : Three Hours Max. Marks: 75

## Section A
**Answer any 10 (1.5 marks each)**

1. Prove that $\{(\xi_1, \xi_2, \xi_3) \in R^3 | \quad \xi_1 = \xi_2 + 1\}$ is not a subspace of $R^3$ but $\{(\xi_1, \xi_2, \xi_3) \in R^3 | \quad \xi_1 = \xi_1 \text{ and } \xi_3 = 0\}$ is a subspace of $R^3$.
2. Define unit sphere in a normed space. Find the unit sphere in the normed space $(X, ||.||)$, where $X$ is the vector space of all ordered pairs of real numbers and $||.||$ is defined by $||x|| = \max\{|\xi_1|, |\xi_2|\}; x = (\xi_1, \xi_2) \in X$ .
3. Define a Schauder basis for a normed space $X$. State an example of a Schauder basis for $l^2$.
4. Define equivalent norms. Give two equivalent norms in $R^2$.
5. Define Hilbert space. Give an example.
6. State and prove Pythagorean theorem in an inner product space.
7. If $X$ is a real inner product space and $x, y \in X$ are such that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ , prove that $x \perp y$.
8. Define the orthogonal complement of a closed subspace $Y$ of a Hilbert space $H$.
9. Show that any linear functional $f$ on $R^3$ can be represented by a dot product.
10. Prove that $||.||$ is a sub linear functional.

**(1.5 x 10 = 15)**

## Section B
**Answer any 4 (5 marks each)**

11. Let $T$ be a linear operator. Then prove that
    a. The range $R(T)$ is a vector space
    b. If dim $D(T) = n < \infty$ , then $\dim R(T) \leq n$
    c. The null space $N(T)$ is a vector space
12. Using the parallelogram equality prove that

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\|z - \frac{1}{2}(x+y)\|^2.$$

13. Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$. Then prove that $\sum_{k=1}^{\infty} \alpha_k e_k$ converges if and only if $\sum_{k=1}^{\infty} |\alpha_k|^2$ is convergent.
14. If $H$ is a separable Hilbert space, then prove that every orthonormal set in $H$ is countable.
15. Prove that every non-zero vector space has a Hamel basis
16. If $X$ is a normed space, $x \in X$ and $g_x$ is a functional defined on $X'$ by $g_x(f) = f(x)$ for all

$f \in X'$, then prove that $g_x$ is bounded linear and $||g_x|| = ||x||$

**(5 x 4 = 20)**

**Section C**
**Answer any 4 (10 marks each)**

17. a. Let $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent set of vectors in a normed space $X$. Then prove that there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$||\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n|| \geq c[|\alpha_1| + \ldots + |\alpha_n|]$$

b. Prove that every finite dimensional subspace $Y$ of a normed space $X$ is complete.

OR

18. a. State and prove $F$ Riesz's Lemma.
b. In a finite dimensional normed space $X$, prove that any set $M \subset X$ is compact if and only if $M$ is closed and bounded.

19. Prove that a metric `$d$' induced by a norm on a normed space $X$ satisfies

a. i. $d(x+a, y+a) = d(x, y)$
ii. $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for all $x, y, a \in X$ and for all scalar $\alpha$.
b. If $d$ is a metric on a vector space $X \neq \{0\}$ which is obtained from a norm, and $\tilde{d}$ is defined by $\tilde{d}(x, x) = 0, \tilde{d}(x, y) = d(x, y) + 1 (x \neq y)$, show that $\tilde{d}$ cannot be obtained from a norm
c. If $c_0$ is the set of all sequences of scalars converging to zero, prove that $c_0$ is a closed subspace of $l^\infty$.

OR

20. a. Let $Y$ be a subspace of a Hilbert space $H$. Then prove that $Y$ is complete if and only if $Y$ is closed in $H$.
b. Prove that $l^p$ with $p \neq 2$ is not an inner product space. Is $l^2$ an inner product space? Justify.
c. Let $T : X \to X$ be a bounded linear operator on a complex inner product space $X$. If $< Tx, x >= 0$ for all $x \in X$, show that $T = 0$.

21. a. An orthogonal set $M$ in a Hilbert space $H$ is total in $H$ if and only if for all $x \in H$, the parseval relation holds.
b. Prove that two Hilbert spaces $H$ and $\tilde{H}$ are isomorphic if and only if they have the same Hilbert dimension.

OR

22. a. Define bounded sesquilinear functional. Give an example.
b. State and prove Reisz representation theorem.

23. a. State and prove Hahn-Banach theorem for a normed space.
b. Let $X$ be a normed space and let $x_0 \neq 0$ be any element of $X$. Then prove that there exists a bounded linear functional $\tilde{f}$ on $X$ such that $||\tilde{f}|| = 1$ and $\tilde{f}(x_0) = ||x_0||$.

OR

24. a. State Baire's category theorem
b. State and prove uniform boundedness theorem.

**(10 x 4 = 40)**