Reg. No .... Name **18P203**

MSc DEGREE END SEMESTER EXAMINATION- APRIL 2018

SEMESTER 2 : MATHEMATICS

COURSE : **16P2MATT06 ; ABSTRACT ALGEBRA**

(Common for Regular - 2017 & Supplementary 2016 admission)

Time : Three Hours Max. Marks: 75

Section A
Answer any 10 (1.5 marks each)

1. What is the largest order among the orders of all the cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$?Justify your answer.
2. Give an example of a subgroup H of a non-abelian group G such that its normalizer $N[H] = G$.
3. Determine the number of abelian groups (upto isomorphism) of order $(10)^5$.
4. Define torsion subgroup of an abelian group. Find the torsion subgroup of the multiplicative group R^* of nonzero real numbers.
5. Find all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[x]/ < x^2 + c >$ is a field?
6. Define simple extension and algebraic extension of a field. Is every finite extension an algebraic extension?
7. Define a constructible number.
8. Let $f(x) \in \mathbb{C}[x]$ have $a + ib$ as a zero. Is $a - ib$ a zero? Justify your answer.
9. What is the order of $G(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$?
10. True or False: $\mathbb{Q}$ is perfect.Justify your answer.

(1.5 x 10 = 15)

Section B
Answer any 4 (5 marks each)

11. Define a decomposable group.Give an example.Show that the finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.
12. Prove that if a finite abelian group has order a power of a prime p, then the order of every element in the group is a power of p. Can the hypothesis of commutativity be dropped? Justify your answer.
13. If F is a field, prove that every non-constant polynomial $f(x) \in F[x]$ can be factored in $F[x]$ into a product of irreducible polynomials, the irreducible polynomials being unique except for order and for unit factors in F.
Find the unique factorization of $x^4 + 3x^3 + 2x + 4$ in $\mathbb{Z}_5[x]$.
14. Prove that if D is an integral domain, then show that $D[x]$ is an integral domain.
15. Show that a finite field of p^n elements has exactly one subfield of p^m elements for each divisor m of n.
16. Show that if E is a finite extension of F, then $\{E : F\}$ divides $[E : F]$.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17. (a). If p and q are distinct primes with $p < q$, show that every group of order pq has a unique subgroup of order q and this subgroup is normal in G.Further show that if $q \not\equiv 1 \,(mod\, p)$, then G is abelian and cyclic.
(b). Show that if H and K are finite subgroups of a group G, then

$$|HK| = \frac{(|H|)(|K|)}{|H \cap K|}.$$

OR

18. (a). Let G be a group containing normal subgroups H and K such that $H \cap K = \{e\}$ and $H \vee K = G$. Show that G is isomorphic to $H \times K$.
(b). Define the class equation of a group G. Using it show that the center of a finite non-trivial p-group G is non-trivial.
(c). Show that a group of order 81 is solvable.

19. (a). Let F be a subfield of a field E, $\alpha \in E$ and x an indeterminate. Show that the map $\phi_\alpha : F[x] \to E$ defined by
$\phi_\alpha(a_0 + a_1x + \ldots + a_nx^n) = a_o + a_1\alpha + \ldots a_n\alpha^n$
is a homomorphism of $F[x]$ into E.
What is this homomorphism known as?
Is this function still a homomorphism if E is a ring and F is a subring of R? Why?
(b). Show that $\sqrt{2}$ is irrational.

OR

20. (a). Show that a finite extension field E of a field F is an algebraic extension of F.
(b). If E is a finite extension of a field F, and K is a finite extension of E, show that K is a finite extension of F, and $[K : F] = [K : E][E : F]$.
(c). If E is a finite extension of a field F, $\alpha \in E$ is algebraic over F, and $\beta \in F(\alpha)$, show that $\deg(\beta, F)$ divides $\deg(\alpha, F)$.

21. Stating the necessary lemmas, establish the existence and uniqueness of $\mathbf{GF}(p^n)$, the Galois field of order p^n.

OR

22. (a). Show that if F is a field of prime characteristic p with algebraic closure $\overline{F}$, then $x^{p^n} - x$ has p^n distinct zeroes in $\overline{F}$.
(b). Show that if F is a field of prime characteristic p, then $(\alpha + \beta)^{p^n} = \alpha^{p^n} + \beta^{p^n}$ for all $\alpha, \beta \in F$ and all positive integers n.

23. Find the splitting field K of $x^4 - 2$ over $\mathbb{Q}$. Compute $G(K/\mathbb{Q})$, find its subgroups and the corresponding fixed fields and draw the subgroup and subfield lattice diagrams.

OR

24. (a). Show that if $f(x)$ is irreducible in $F[x]$, then it has a factorization of the form $a\prod_i (x - \alpha_i)^\nu$ in $\overline{F}[x]$.
(b). Give an example of an irreducible polynomial having a zero of multiplicity greater than 1.

(10 x 4 = 40)