Reg. No .... Name **18P131**

M.Sc DEGREE END SEMESTER EXAMINATION - NOVEMBER 2018

SEMESTER 1 : MATHEMATICS

COURSE : **16P1MATT03 : MEASURE THEORY AND INTEGRATION**

(For Regular - 2018 Admission & Supplementary - 2016 / 2017 Admissions)

Time : Three Hours Max. Marks: 75

Section A

Answer any 10 (1.5 marks each)

1. Define an F_σ-set and a G_δ-set. Are they measurable? Justify.
2. Given that (a, ∞) is Lebesgue measurable for any $a \in R$. Using this property prove that any finite interval (a, b) is measurable.
3. Let $E \subset M$ and M be measurable with $m(M) < \infty$. If E is measurable, show that
$$m(M) = m^*E + m^*(M - E).$$
4. If A and B are two disjoint measurable sets of finite measure and f is a bounded measurable function, then prove that
$$\int_{A \cup B} f = \int_A f + \int_B f.$$
5. Let f be a bounded measurable function defined on a measurable set E of finite measure.
If A and B are two constants and $A \leq f \leq B$, then prove that $AmE \leq \int_E f \leq BmE$.
6. If f is integrable, then prove that f is finite valued a.e.
7. Suppose $\langle f_n \rangle$ is a sequence of measurable functions on the measurable space $(X, \mathcal{B})$. Then prove that $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are measurable.
8. Define μ^*-measurable set. Prove that ϕ is μ^*-measurable.
9. If μ is a measure on an algebra $\mathcal{A}$ and μ^* is the outer measure defined by μ, prove that $\mu^* A = \mu A$ if $A \in \mathcal{A}$.
10. Define the x-section and y-section of a set $E \subset X \times Y$.

(1.5 x 10 = 15)

Section B

Answer any 4 (5 marks each)

11. (a) Show that every non-empty open set has a positive measure.
(b) If $\{A_n\}$ is a countable collection of sets of real numbers, then prove that
$$m^*(\cup A_n) \leq \sum m^* A_n.$$
Can m^* be finitely subadditive?. Justify.
12. For $k > 0$ and $A \subset R$, let $kA = \{kx : x \in A\}$ show that,
(i) $m^*(kA) = km^*A$ and
(ii) A is measurable if and only if kA is measurable.

13. Let $\langle u_n \rangle$ be a sequence of non-negative measurable functions and let $f = \sum_1^\infty u_n$. Then prove that $\int f = \sum_1^\infty \int u_n$.

14. Let f be a non-negative measurable function and $\langle E_i \rangle$ be a disjoint sequence of measurable sets. Let $E = \cup E_i$. Then prove that

$$\int_E f = \sum \int_{E_i} f.$$

15. Let $\mathcal{A}$ be an algebra of subsets of a space X. If $A \in \mathcal{A}$ and if $\langle A_i \rangle$ is any sequence of sets in $\mathcal{A}$ such that $A \subset \bigcup_{i=1}^\infty A_i$, then prove that

$$\mu A \leq \sum_{i=1}^\infty \mu A_i.$$

16. Prove that $\mathcal{S} \times \mathcal{J} = \mathcal{S}(\mathcal{E})$, the σ-algebra generated by $\mathcal{E}$.

(5 x 4 = 20)

Section C

Answer any 4 (10 marks each)

17.1. (a) Prove that the outer measure of an interval is its length.
(b) Let A be the set of all rational numbers between 0 and 1 and let $\{I_n\}$ be a finite collection of open intervals covering A. Then prove that $\sum l(I_n) \geq 1$.

OR

2. (a) If f and g are two real valued measurable functions with the same domain, then
(i) Prove that $f + g$ is measurable.
(ii) Prove that cf is measurable, if c is a constant. Hence prove that $af + bg$ is measurable, if a and b are two constants. Deduce that $f - g$ is measurable.
(b) If f is a real valued measurable function defined on $(-\infty, \infty)$ and g is a continuous function, then prove that $g \circ f$ is measurable.

18.1. (a) Define Riemann integral of a bounded function over a finite closed integral $[a, b]$ interms of step functions.
(b) Define Lebesgue integral of a bounded measurable function defined on a measurable set E with mE finite.
(c) Let f be a bounded function defined an $[a, b]$. If f is Riemann integrable, then prove that it is measurable and

$$R \int_a^b f(x)dx = \int_a^b f(x)dx.$$

OR

2. (a) Let $\langle g_n \rangle$ be a sequence of integrable functions which converges a.e. to an integrable function g. Let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\langle f_n \rangle$ converges to f a.e. If $\int g = \lim \int g_n$, prove that

$$\int f = \lim \int f_n.$$

(b) Show that if f is integrable over E, then so is $|f|$ and $|\int_E f| \leq \int_E |f|$. Does the integrability of $|f|$ imply that of f? Justify.

19.1. (a) State and prove Hahn decomposition theorem.
(b) Give an example to show that the Hahn decomposition need not be unique.

OR

2. (a) State and prove Jordan decomposition theorem.
(b) Let E be a measurable set such that $0 < \nu E < \infty$. Then prove that there is a positive set $A \subset E$ with $\nu A > 0$.

20.1. If $E \in \mathcal{S} \times \mathcal{J}$, then prove that for each $x \in X$ and $y \in Y$, $E_x \in \mathcal{J}$ and $E^y \in \mathcal{S}$.

OR

2. Let $[[X, \mathcal{S}, \mu]]$ and $[[Y, \mathcal{J}, v]]$ be σ-finite measure spaces. For $V \in \mathcal{S} \times \mathcal{J}$, write $\phi(x) = \nu(V_x)$ and $\psi(y) = \mu(V^y)$ for all $x \in X$ and $y \in Y$. Then prove that ϕ is $\mathcal{S}$-measurable and ψ is $\mathcal{J}$-measurable and $\int_X \phi d\mu = \int_Y \psi d\nu$.

(10 x 4 = 40)